Multiple Integrals

Double and Triple Integrals, Change of Order, and Variables

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Outline

Double Integration: Definition

A double integral over a region R in the xy-plane is defined as:

$$\iint_R f(x,y)\,dA = \lim_{\Delta A \to 0} \sum f(x_i, y_i)\Delta A$$

- In **Cartesian coordinates**: $dA = dx\,dy$
- In **polar coordinates**: $dA = r\,dr\,d\theta$

Double Integration: Cartesian Form

Method of Evaluation (Cartesian):

- Identify the region R defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$.
- Compute: $\iint_R f(x, y)\, dx\, dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$.

Double Integration: Cartesian Form

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- Compute: $\iint_R f(x,y)\,dx\,dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$.

Examples:

1. Compute $\iint_R 1\,dA$, where R is the rectangle $[0,1] \times [0,2]$.

$$\iint_R 1\,dx\,dy = \int_0^1 \int_0^2 1\,dy\,dx = \int_0^1 [y]_0^2\,dx = \int_0^1 2\,dx = 2$$

2. Compute $\iint_R x\,dA$, where $R = [0,1] \times [0,1]$.

$$\int_0^1 \int_0^1 x\,dy\,dx = \int_0^1 x[y]_0^1\,dx = \int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

3. Compute $\iint_R xy\,dA$, where $R = [0,2] \times [0,1]$.

$$\int_0^2 \int_0^1 xy\,dy\,dx = \int_0^2 x\left[\frac{y^2}{2}\right]_0^1\,dx = \int_0^2 \frac{x}{2}\,dx = \left[\frac{x^2}{4}\right]_0^2 = 1$$

Double Integration: Cartesian Examples (Cont.)

4. Compute $\iint_R (x^2 + y^2)\, dA$, where $R = [0,1] \times [0,1]$.

$$\int_0^1 \int_0^1 (x^2+y^2)\, dy\, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^1 dx = \int_0^1 \left(x^2 + \frac{1}{3}\right) dx = \left[\frac{x^3}{3}\right.$$

5. Compute $\iint_R e^{x+y}\, dA$, where $R = [0,1] \times [0,1]$.

$$\int_0^1 \int_0^1 e^{x+y}\, dy\, dx = \int_0^1 e^x [e^y]_0^1\, dx = \int_0^1 e^x (e-1)\, dx = (e-1)[e^x]_0^1 = (e$$

6. Compute $\iint_R x^2 y\, dA$, where R is the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$.

$$\int_0^1 \int_0^{1-x} x^2 y\, dy\, dx = \int_0^1 x^2 \left[\frac{y^2}{2}\right]_0^{1-x} dx = \int_0^1 \frac{x^2(1-x)^2}{2}\, dx = \frac{1}{2}\int_0^1 (x$$

Double Integration: Cartesian Examples (Cont.)

7. Compute $\iint_R \sin(x+y)\,dA$, where $R=[0,\pi]\times[0,\pi]$.

$$\int_0^\pi\int_0^\pi \sin(x+y)\,dy\,dx = \int_0^\pi [-\cos(x+y)]_0^\pi\,dx = \int_0^\pi (-\cos(x+\pi)+\cos x$$

8. Compute $\iint_R xy^2\,dA$, where $R=[1,2]\times[0,1]$.

$$\int_1^2\int_0^1 xy^2\,dy\,dx = \int_1^2 x\left[\frac{y^3}{3}\right]_0^1 dx = \int_1^2 \frac{x}{3}\,dx = \frac{3}{2}$$

9. Compute $\iint_R (x+y)\,dA$, where R is the region bounded by $y=x^2$, $y=1$.

$$\int_0^1\int_{x^2}^1 (x+y)\,dy\,dx = \int_0^1 \left[xy+\frac{y^2}{2}\right]_{x^2}^1 dx = \int_0^1 \left((x+\frac{1}{2})-(x^3+\frac{x^4}{2})\right)$$

10. Compute $\iint_R y\,dA$, where $R=[0,1]\times[0,2]$.

$$\int_0^1\int_0^2 y\,dy\,dx = \int_0^1 \left[\frac{y^2}{2}\right]_0^2 dx = \int_0^1 2\,dx = 2$$

Double Integration: Polar Form

Method of Evaluation (Polar):

- Transform (x, y) to (r, θ): $x = r\cos\theta$, $y = r\sin\theta$, $dA = r\,dr\,d\theta$.
- Set up integral over region R defined by r and θ bounds.

Double Integration: Polar Form

Method of Evaluation (Polar):

- Transform (x, y) to (r, θ): $x = r\cos\theta$, $y = r\sin\theta$, $dA = r\,dr\,d\theta$.
- Set up integral over region R defined by r and θ bounds.

Examples:

1. Compute $\iint_R 1\,dA$, where R is the unit disk $x^2 + y^2 \leq 1$.

$$\int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^1 d\theta = \int_0^{2\pi}\frac{1}{2}\,d\theta = \pi$$

2. Compute $\iint_R (x^2 + y^2)\,dA$, where R is the unit disk.

$$\int_0^{2\pi}\int_0^1 r^2 \cdot r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^1 d\theta = \int_0^{2\pi}\frac{1}{4}\,d\theta = \frac{\pi}{2}$$

3. Compute $\iint_R x^2\,dA$, where R is the unit disk.

$$\int_0^{2\pi}\int_0^1 (r^2\cos^2\theta) r\,dr\,d\theta = \int_0^{2\pi}\cos^2\theta\left[\frac{r^4}{4}\right]_0^1 d\theta = \frac{1}{4}\int_0^{2\pi}\cos^2\theta\,d\theta = \frac{\pi}{4}$$

Double Integration: Polar Examples (Cont.)

4. Compute $\iint_R e^{-(x^2+y^2)}\, dA$, where R is the entire plane.

$$\int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\, dr\, d\theta = \int_0^{2\pi}\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} d\theta = \int_0^{2\pi}\frac{1}{2}\, d\theta = \pi$$

5. Compute $\iint_R y\, dA$, where R is the unit disk.

$$\int_0^{2\pi}\int_0^{1}(r\sin\theta)r\, dr\, d\theta = \int_0^{2\pi}\sin\theta\left[\frac{r^3}{3}\right]_0^{1} d\theta = \frac{1}{3}\int_0^{2\pi}\sin\theta\, d\theta = 0$$

6. Compute $\iint_R xy\, dA$, where R is the disk $x^2+y^2 \leq 4$.

$$\int_0^{2\pi}\int_0^{2}(r^2\cos\theta\sin\theta)r\, dr\, d\theta = \int_0^{2\pi}\cos\theta\sin\theta\left[\frac{r^4}{4}\right]_0^{2} d\theta = 4\int_0^{2\pi}\frac{\sin 2\theta}{2}\, d$$

Double Integration: Polar Examples (Cont.)

7. Compute $\iint_R (x^2+y^2)^{3/2}\,dA$, where R is the annulus $1 \le x^2+y^2 \le 4$.

$$\int_0^{2\pi}\int_1^2 r^3 \cdot r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^5}{5}\right]_1^2 d\theta = \frac{2\pi}{5}(32-1) = \frac{62\pi}{5}$$

8. Compute $\iint_R \cos(x^2+y^2)\,dA$, where R is the disk $x^2+y^2 \le \pi$.

$$\int_0^{2\pi}\int_0^{\sqrt{\pi}} \cos(r^2) r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{\sin(r^2)}{2}\right]_0^{\sqrt{\pi}} d\theta = \pi\sin(\pi) = 0$$

9. Compute $\iint_R x^4\,dA$, where R is the unit disk.

$$\int_0^{2\pi}\int_0^1 (r^4\cos^4\theta) r\,dr\,d\theta = \int_0^{2\pi}\cos^4\theta\left[\frac{r^6}{6}\right]_0^1 d\theta = \frac{1}{6}\cdot\frac{3\pi}{4} = \frac{\pi}{8}$$

10. Compute $\iint_R \sqrt{x^2+y^2}\,dA$, where R is the disk $x^2+y^2 \le 1$.

$$\int^{2\pi}\int^{1} r\cdot r\,dr\,d\theta = \int^{2\pi}\left[\frac{r^3}{3}\right]^1 d\theta = \frac{2\pi}{3}$$

Change of Order of Integration

For a double integral $\iint_R f(x,y)\,dA$, changing the order of integration involves redefining the region R in terms of the opposite variable bounds.

- Original: $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$
- Changed: $\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$

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Examples:

1. Evaluate $\int_0^1 \int_0^x xy\, dy\, dx$.

$$\int_0^1 \int_0^x xy\, dy\, dx = \int_0^1 x \left[\frac{y^2}{2}\right]_0^x dx = \int_0^1 \frac{x^3}{2}\, dx = \frac{1}{8}$$

Change order ($0 \le y \le 1$, $y \le x \le 1$):

$$\int_0^1 \int_y^1 xy\, dx\, dy = \int_0^1 y \left[\frac{x^2}{2}\right]_y^1 dy = \int_0^1 \frac{y(1-y^2)}{2}\, dy = \frac{1}{8}$$

Change of Order Examples (Cont.)

2. Evaluate $\int_0^2 \int_0^{y/2} x\,dx\,dy$.

$$\int_0^2 \int_0^{y/2} x\,dx\,dy = \int_0^2 \left[\frac{x^2}{2}\right]_0^{y/2} dy = \int_0^2 \frac{y^2}{8}\,dy = \frac{2}{3}$$

Change order ($0 \le x \le 1$, $2x \le y \le 2$):

$$\int_0^1 \int_{2x}^2 x\,dy\,dx = \int_0^1 x(2-2x)\,dx = \frac{2}{3}$$

3. Evaluate $\int_0^1 \int_y^1 e^{x^2}\,dx\,dy$.

Change order ($0 \le x \le 1$, $0 \le y \le x$): $\int_0^1 \int_0^x e^{x^2}\,dy\,dx = \int_0^1 e^{x^2} x\,dx =$

4. Evaluate $\int_0^1 \int_0^{1-x} xy\,dy\,dx$.

$$\int_0^1 \int_0^{1-x} xy\,dy\,dx = \frac{1}{24}$$

Change order ($0 \le y \le 1$, $0 \le x \le 1-y$):

Change of Order Examples (Cont.)

5. Evaluate $\int_0^1 \int_{y^2}^1 \sqrt{x}\, dx\, dy$.

$$\int_0^1 \int_{y^2}^1 x^{1/2}\, dx\, dy = \frac{2}{5}$$

Change order ($0 \le x \le 1$, $0 \le y \le \sqrt{x}$):

$$\int_0^1 \int_0^{\sqrt{x}} x^{1/2}\, dy\, dx = \frac{2}{5}$$

6. Evaluate $\int_0^2 \int_{y/2}^1 y\, dx\, dy$.

$$\int_0^2 \int_{y/2}^1 y\, dx\, dy = \int_0^2 y\left(1 - \frac{y}{2}\right)\, dy = 1$$

Change order ($0 \le x \le 1$, $0 \le y \le 2x$):

$$\int_0^1 \int_0^{2x} y\, dy\, dx = 1$$

7. Evaluate $\int^{\pi} \int^{\sin x} y\, dy\, dx$

Change of Order Examples (Cont.)

8. Evaluate $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$.

   Change order $(0 \le y \le 1,\ 0 \le x \le y)$: $\displaystyle\int_0^1 \int_0^y \sin(y^2)\, dx\, dy = \frac{1 - \cos 1}{2}$

9. Evaluate $\int_0^1 \int_0^{1-x^2} xy\, dy\, dx$.

$$\int_0^1 \int_0^{1-x^2} xy\, dy\, dx = \frac{1}{12}$$

   Change order $(0 \le y \le 1,\ 0 \le x \le \sqrt{1-y})$:

$$\int_0^1 \int_0^{\sqrt{1-y}} xy\, dx\, dy = \frac{1}{12}$$

10. Evaluate $\int_0^1 \int_{x^2}^{x} xy\, dy\, dx$.

$$\int_0^1 \int_{x^2}^{x} xy\, dy\, dx = \frac{1}{24}$$

Change of Variables

To evaluate $\iint_R f(x,y)\,dx\,dy$, transform to new variables (u,v) via $x = x(u,v)$, $y = y(u,v)$:

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

where $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ is the Jacobian determinant.

Change of Variables

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where $\left| \frac{\partial(x,y)}{\partial(u,v)} \right|$ is the Jacobian determinant. **Examples**:

1. Evaluate $\iint_R (x+y)\,dA$, where R is the parallelogram with vertices $(0,0)$, $(1,1)$, $(2,0)$, $(1,-1)$. Use $u = x+y$, $v = x-y$.

$$x = \frac{u+v}{2}, \quad y = \frac{u-v}{2}, \quad \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{2}$$

Region S: $0 \le u \le 2$, $-1 \le v \le 1$.

$$\int_0^2 \int_{-1}^1 u \cdot \frac{1}{2}\,dv\,du = \int_0^2 u\,du = 2$$

Change of Variables Examples (Cont.)

2. Evaluate $\iint_R xy\,dA$, where R is the region bounded by $xy = 1$, $xy = 2$, $y = x$, $y = 2x$. Use $u = xy$, $v = y/x$.

$$x = \sqrt{u/v}, \quad y = \sqrt{uv}, \quad \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2\sqrt{uv}}$$

   Region: $1 \le u \le 2$, $1 \le v \le 2$.

$$\int_1^2 \int_1^2 u \cdot \frac{1}{2\sqrt{uv}}\,dv\,du = \frac{\ln 2}{2}$$

3. Evaluate $\iint_R (x^2 - y^2)\,dA$, where R is the unit square, using $u = x + y$, $v = x - y$.

$$\int_0^2 \int_{1-u}^{u-1} (u^2 - v^2) \cdot \frac{1}{2}\,dv\,du = 0$$

4. Evaluate $\iint_R \sqrt{x^2 + y^2}\,dA$, where R is the unit disk, using polar coordinates.

$$\int^{2\pi} \int^{1} r \cdot r\,dr\,d\theta = \frac{2\pi}{3}$$

Change of Variables Examples (Cont.)

5. Evaluate $\iint_R e^{x/y}\, dA$, where R is bounded by $y = x$, $y = 2x$, $x = 1$, $x = 2$. Use $u = x/y$, $v = y$.

$$\int_1^2 \int_v^{2v} e^{x/v} \cdot \frac{1}{v}\, dx\, dv = 2(e^2 - e)$$

6. Evaluate $\iint_R x\, dA$, where R is the region $x^2 + y^2 \leq 1$, using polar coordinates.

$$\int_0^{2\pi} \int_0^1 (r \cos\theta) r\, dr\, d\theta = 0$$

7. Evaluate $\iint_R (x + y)^2\, dA$, where R is the square $[0, 1] \times [0, 1]$, using $u = x + y$, $v = x - y$.

$$\int_0^2 \int_{1-u}^{u-1} u^2 \cdot \frac{1}{2}\, dv\, du = \frac{2}{3}$$

8. Evaluate $\iint_R \sin(x/y)\, dA$, where R is bounded by $y = x$, $y = 2x$, $x = 0$, $x = 1$. Use $u = x/y$, $v = y$.

Change of Variables Examples (Cont.)

9. Evaluate $\iint_R x^2y^2\,dA$, where R is bounded by $xy = 1$, $xy = 2$, $y = x$, $y = 2x$. Use $u = xy$, $v = y/x$.

$$\int_1^2 \int_1^2 u^2 \cdot \frac{1}{2\sqrt{uv}}\,dv\,du = \frac{2\ln 2}{3}$$

10. Evaluate $\iint_R (x^2 + y^2)\,dA$, where R is the disk $x^2 + y^2 \leq 4$, using polar coordinates.

$$\int_0^{2\pi} \int_0^2 r^2 \cdot r\,dr\,d\theta = 8\pi$$

Triple Integrals

A triple integral over a region W in 3D space is:

$$\iiint_W f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

where $dV = dx\,dy\,dz$ in Cartesian coordinates.

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where $dV = dx\,dy\,dz$ in Cartesian coordinates. **Examples**:

1. Compute $\iiint_W 1\,dV$, where W is the cube $[0,1] \times [0,1] \times [0,1]$.

$$\int_0^1 \int_0^1 \int_0^1 1\,dz\,dy\,dx = 1$$

2. Compute $\iiint_W x\,dV$, where W is the cube $[0,1] \times [0,1] \times [0,1]$.

$$\int_0^1 \int_0^1 \int_0^1 x\,dz\,dy\,dx = \frac{1}{2}$$

3. Compute $\iiint_W xyz\,dV$, where W is the cube $[0,1] \times [0,1] \times [0,1]$.

$$\int_0^1 \int_0^1 \int_0^1 xyz\,dz\,dy\,dx = \frac{1}{8}$$

Triple Integral Examples (Cont.)

4. Compute $\iiint_W z\,dV$, where W is the tetrahedron bounded by $x=0$, $y=0$, $z=0$, $x+y+z=1$.

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx = \frac{1}{24}$$

5. Compute $\iiint_W (x^2+y^2)\,dV$, where W is the unit cube.

$$\int_0^1 \int_0^1 \int_0^1 (x^2+y^2)\,dz\,dy\,dx = \frac{2}{3}$$

6. Compute $\iiint_W e^z\,dV$, where W is the region $0 \le z \le 1$, $0 \le y \le z$, $0 \le x \le y$.

$$\int_0^1 \int_0^z \int_0^y e^z\,dx\,dy\,dz = \frac{e-1}{2}$$

Triple Integral Examples (Cont.)

7. Compute $\iiint_W x^2\, dV$, where W is the region bounded by $z = x^2 + y^2$, $z = 1$.

$$\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{x^2+y^2}^{1} x^2\, dz\, dy\, dx = \frac{\pi}{12}$$

8. Compute $\iiint_W (x + y + z)\, dV$, where W is the tetrahedron $x, y, z \geq 0$, $x + y + z \leq 1$.

$$\int_{0}^{1} \int_{0}^{1-x} \int_{0}^{1-x-y} (x + y + z)\, dz\, dy\, dx = \frac{1}{8}$$

9. Compute $\iiint_W z^2\, dV$, where W is the unit cube.

$$\int_{0}^{1} \int_{0}^{1} \int_{0}^{1} z^2\, dz\, dy\, dx = \frac{1}{3}$$

10. Compute $\iiint_W \sqrt{x^2 + y^2 + z^2}\, dV$, where W is the unit ball $x^2 + y^2 + z^2 \leq 1$ (use spherical coordinates).

Conclusion

- Double integrals can be evaluated in Cartesian or polar coordinates, depending on the region.
- Changing the order of integration simplifies computations by adjusting region bounds.
- Change of variables, including polar and other transformations, leverages the Jacobian.
- Triple integrals extend these concepts to 3D regions, with applications in volume and mass calculations.