Laplace and Fourier Transforms

Definitions, Properties, and Applications

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Outline

Laplace Transform: Definitions

The **Laplace transform** of a function $f(t)$, $t \geq 0$, is:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt, \quad \Re(s) > \sigma$$

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- **Piecewise continuity**: $f(t)$ is continuous except at finitely many points in any finite interval.
- **Exponential order**: There exist $M > 0$, σ such that $|f(t)| \leq Me^{\sigma t}$ for large t.
- **Class A**: Piecewise continuous and of exponential order.

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- **Class A**: Piecewise continuous and of exponential order.

Examples:

1. $f(t) = 1$: Piecewise continuous, exponential order $\sigma = 0$.
2. $f(t) = e^{at}$: Exponential order $\sigma = a$.
3. $f(t) = \sin at$: Exponential order $\sigma = 0$.
4. $f(t) = t^n$: Exponential order $\sigma = 0$.
5. $f(t) = u(t - a)$ (unit step): Piecewise continuous, $\sigma = 0$.
6. $f(t) = te^{2t}$: Exponential order $\sigma = 2$.
7. $f(t) = \cos at$: Exponential order $\sigma = 0$.
8. $f(t) = t^2 \sin t$: Exponential order $\sigma = 0$.
9. $f(t) = e^{-t} t$: Exponential order $\sigma = -1$.

Existence Theorem and Standard Transforms

Existence Theorem: If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order σ, then $\mathcal{L}\{f(t)\}$ exists for $\Re(s) > \sigma$.

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1. $\mathcal{L}\{1\} = \frac{1}{s}$, $s > 0$
2. $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, $s > 0$
3. $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, $s > a$
4. $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$, $s > 0$
5. $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$, $s > 0$
6. $\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$, $s > |a|$
7. $\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$, $s > |a|$

Shifting Theorems

First Shifting Theorem:

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

Second Shifting Theorem:

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s), \quad u(t) = \text{unit step}$$

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Examples:

1. $\mathcal{L}\{e^{3t}t^2\} = \frac{2}{(s-3)^3}$
2. $\mathcal{L}\{e^{-2t}\sin 4t\} = \frac{4}{(s+2)^2+16}$
3. $\mathcal{L}\{(t-2)^2 u(t-2)\} = e^{-2s}\frac{2}{s^3}$
4. $\mathcal{L}\{e^{t}\cos t\} = \frac{s-1}{(s-1)^2+1}$
5. $\mathcal{L}\{\sin(t-\pi)u(t-\pi)\} = e^{-\pi s}\frac{1}{s^2+1}$
6. $\mathcal{L}\{e^{4t}t^3\} = \frac{6}{(s-4)^4}$
7. $\mathcal{L}\{(t-1)u(t-1)\} = e^{-s}\frac{1}{s^2}$
8. $\mathcal{L}\{e^{-t}\cosh 2t\} = \frac{s+1}{(s+1)^2-4}$
9. $\mathcal{L}\{te^{2t}\} = \frac{1}{(s-2)^2}$
10. $\mathcal{L}\{\cos(2(t-1))u(t-1)\} = e^{-s}\frac{s}{s^2+4}$

Change of Scale Property

$$\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right), \quad a > 0$$

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Examples:

1. $\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}$
2. $\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}$
3. $\mathcal{L}\{t^2 e^{5t}\} = \frac{2}{(s-5)^3}$
4. $\mathcal{L}\{t \sin 4t\} = \frac{8s}{(s^2+16)^2}$
5. $\mathcal{L}\{e^{2t} t^3\} = \frac{6}{(s-2)^4}$
6. $\mathcal{L}\{\sin(5t)\} = \frac{5}{s^2+25}$
7. $\mathcal{L}\{t^2 \cos t\} = \frac{2(s^2-1)}{(s^2+1)^3}$
8. $\mathcal{L}\{\cosh 3t\} = \frac{s}{s^2-9}$
9. $\mathcal{L}\{te^t\} = \frac{1}{(s-1)^2}$
10. $\mathcal{L}\{\sin(2t/3)\} = \frac{2/3}{s^2+(2/3)^2} = \frac{2}{3s^2+4}$

Laplace Transform of Derivatives

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0), \quad \mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

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Examples (assume $f(0) = 0$, $f'(0) = 0$ unless specified):

1. $\mathcal{L}\{t'\} = s \cdot \frac{1}{s^2} = \frac{1}{s}$
2. $\mathcal{L}\{\sin'(t)\} = s \cdot \frac{1}{s^2+1} = \frac{s}{s^2+1}$
3. $\mathcal{L}\{e^{2t\prime}\} = s \cdot \frac{1}{s-2} - 1$
4. $\mathcal{L}\{t^{2\prime\prime}\} = s^2 \cdot \frac{2}{s^3} = \frac{2}{s}$
5. $\mathcal{L}\{\cos'(t)\} = s \cdot \frac{s}{s^2+1} = \frac{s^2}{s^2+1}$
6. $\mathcal{L}\{(te^t)'\} = s \cdot \frac{1}{(s-1)^2} - 0$
7. $\mathcal{L}\{\sin(2t)''\} = s^2 \cdot \frac{2}{s^2+4}$
8. $\mathcal{L}\{(t^3)'\} = s \cdot \frac{6}{s^4} = \frac{6}{s^3}$
9. $\mathcal{L}\{e^{-t\prime}\} = s \cdot \frac{1}{s+1} - 1$
10. $\mathcal{L}\{(t^2e^{2t})'\} = s \cdot \frac{2}{(s-2)^3}$

Laplace Transform of Integrals

$$\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{F(s)}{s}$$

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Examples:

1. $\mathcal{L}\left\{\int_0^t 1\,du\right\} = \frac{1/s}{s} = \frac{1}{s^2}$
2. $\mathcal{L}\left\{\int_0^t u\,du\right\} = \frac{1/s^2}{s} = \frac{1}{s^3}$
3. $\mathcal{L}\left\{\int_0^t \sin u\,du\right\} = \frac{1/(s^2+1)}{s} = \frac{1}{s(s^2+1)}$
4. $\mathcal{L}\left\{\int_0^t e^u\,du\right\} = \frac{1/(s-1)}{s} = \frac{1}{s(s-1)}$
5. $\mathcal{L}\left\{\int_0^t u^2\,du\right\} = \frac{2/s^3}{s} = \frac{2}{s^4}$
6. $\mathcal{L}\left\{\int_0^t \cos 2u\,du\right\} = \frac{s/(s^2+4)}{s} = \frac{1}{s^2+4}$
7. $\mathcal{L}\left\{\int_0^t ue^u\,du\right\} = \frac{1/(s-1)^2}{s}$
8. $\mathcal{L}\left\{\int_0^t \sinh u\,du\right\} = \frac{1/(s^2-1)}{s}$
9. $\mathcal{L}\left\{\int_0^t u\sin u\,du\right\} = \frac{1/(s^2+1)^2}{s}$
10. $\mathcal{L}\left\{\int_0^t e^{2u}\,du\right\} = \frac{1/(s-2)}{}$

Multiplication by t^n

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Examples:

1. $\mathcal{L}\{t \sin t\} = -\frac{d}{ds}\frac{1}{s^2+1} = \frac{2s}{(s^2+1)^2}$
2. $\mathcal{L}\{t^2 e^t\} = \frac{d^2}{ds^2}\frac{1}{s-1} = \frac{2}{(s-1)^3}$
3. $\mathcal{L}\{t \cos t\} = -\frac{d}{ds}\frac{s}{s^2+1} = \frac{s^2-1}{(s^2+1)^2}$
4. $\mathcal{L}\{t^3 t\} = -\frac{d^3}{ds^3}\frac{1}{s^2} = \frac{24}{s^5}$
5. $\mathcal{L}\{te^{2t}\} = -\frac{d}{ds}\frac{1}{s-2} = \frac{1}{(s-2)^2}$
6. $\mathcal{L}\{t^2 \sin 2t\} = \frac{d^2}{ds^2}\frac{2}{s^2+4}$
7. $\mathcal{L}\{t \sinh t\} = -\frac{d}{ds}\frac{1}{s^2-1} = \frac{2s}{(s^2-1)^2}$
8. $\mathcal{L}\{t^2 \cos t\} = \frac{d^2}{ds^2}\frac{s}{s^2+1}$
9. $\mathcal{L}\{te^{-t}\} = \frac{1}{(s+1)^2}$
10. $\mathcal{L}\{t^3 e^{3t}\} = \frac{6}{(s-3)^4}$

Division by t

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_{s}^{\infty} F(u)\, du$$

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Examples:

1. $\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{u^2+1}\,du = \arctan\frac{1}{s}$
2. $\mathcal{L}\left\{\frac{e^t}{t}\right\} = \int_s^\infty \frac{1}{u-1}\,du = \ln\frac{s-1}{s}$
3. $\mathcal{L}\left\{\frac{\cos t}{t}\right\} = \int_s^\infty \frac{u}{u^2+1}\,du$
4. $\mathcal{L}\left\{\frac{t}{t}\right\} = \int_s^\infty \frac{1}{u^2}\,du = \frac{1}{s}$
5. $\mathcal{L}\left\{\frac{\sin 2t}{t}\right\} = \arctan\frac{2}{s}$
6. $\mathcal{L}\left\{\frac{e^{2t}}{t}\right\} = \ln\frac{s-2}{s}$
7. $\mathcal{L}\left\{\frac{\sinh t}{t}\right\} = \int_s^\infty \frac{1}{u^2-1}\,du$
8. $\mathcal{L}\left\{\frac{\cos 2t}{t}\right\} = \int_s^\infty \frac{u}{u^2+4}\,du$
9. $\mathcal{L}\left\{\frac{t^2}{t}\right\} = \frac{2}{s^3}$
10. $\mathcal{L}\left\{\frac{e^{-t}}{t}\right\} = \ln\frac{s+1}{s}$

Inverse Laplace Transform: Definition

The **inverse Laplace transform** is:

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Standard Inverse Transforms:

1. $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
2. $\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$
3. $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
4. $\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$
5. $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$
6. $\mathcal{L}^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at$
7. $\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

Inverse Laplace Examples

Examples:

1. $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$
2. $\mathcal{L}^{-1}\left\{\frac{2}{s-3}\right\} = 2e^{3t}$
3. $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t$
4. $\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} = \cos 3t$
5. $\mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\} = t^3$
6. $\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = \sinh t$
7. $\mathcal{L}^{-1}\left\{\frac{s}{s^2-4}\right\} = \cosh 2t$
8. $\mathcal{L}^{-1}\left\{\frac{3}{s+2}\right\} = 3e^{-2t}$
9. $\mathcal{L}^{-1}\left\{\frac{2}{s^2+1}\right\} = 2\sin t$
10. $\mathcal{L}^{-1}\left\{\frac{24}{s^5}\right\} = t^4$

Inverse Shifting Theorems

First Shifting Theorem:

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$$

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Examples:

1. $\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\} = te^{2t}$
2. $\mathcal{L}^{-1}\left\{e^{-3s}\frac{1}{s^2}\right\} = (t-3)u(t-3)$
3. $\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\} = e^{-t}\sin 2t$
4. $\mathcal{L}^{-1}\left\{e^{-s}\frac{s}{s^2+1}\right\} = \cos(t-1)u(t-1)$
5. $\mathcal{L}^{-1}\left\{\frac{3}{(s-4)^3}\right\} = \frac{3t^2e^{4t}}{2}$
6. $\mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s}\right\} = u(t-2)$
7. $\mathcal{L}^{-1}\left\{\frac{s}{(s-1)^2+9}\right\} = e^{t}\cos 3t$
8. $\mathcal{L}^{-1}\left\{e^{-4s}\frac{2}{s^3}\right\} = (t-4)^2u(t-4)$
9. $\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\} = te^{-3t}$

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Examples (assume $f(0) = 0$):

1. $\mathcal{L}^{-1}\left\{s \cdot \frac{1}{s^2}\right\} = 1$
2. $\mathcal{L}^{-1}\left\{s \cdot \frac{1}{s^2+1}\right\} = \cos t$
3. $\mathcal{L}^{-1}\left\{s \cdot \frac{1}{s-1}\right\} = e^t$
4. $\mathcal{L}^{-1}\left\{s \cdot \frac{2}{s^2+4}\right\} = 2\cos 2t$
5. $\mathcal{L}^{-1}\left\{s \cdot \frac{1}{(s-2)^2}\right\} = (2t+1)e^{2t}$
6. $\mathcal{L}^{-1}\left\{s \cdot \frac{6}{s^4}\right\} = 3t^2$
7. $\mathcal{L}^{-1}\left\{s \cdot \frac{s}{s^2+9}\right\} = 3\cos 3t$
8. $\mathcal{L}^{-1}\left\{s \cdot \frac{1}{s+1}\right\} = e^{-t}$
9. $\mathcal{L}^{-1}\left\{s \cdot \frac{2}{(s-3)^3}\right\} = 3t^2 e^{3t}$
10. $\mathcal{L}^{-1}\left\{s \cdot \frac{1}{s^2-1}\right\} = \cosh t$

Convolution Theorem

$$\mathcal{L}\{f(t) * g(t)\} = F(s)G(s), \quad f * g = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

Multiplication by s:

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Convolution Examples:

1. $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{s}\right\} = \int_0^t \tau \cdot 1\,d\tau = \frac{t^2}{2}$
2. $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\cdot\frac{1}{s^2+1}\right\} = \int_0^t \sin\tau\sin(t-\tau)\,d\tau = \frac{1}{2}(\sin t - t\cos t)$
3. $\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s-1}\right\} = \int_0^t 1\cdot e^{t-\tau}\,d\tau = e^t - 1$
4. $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{2}{s^2+4}\right\} = \int_0^t \tau\sin 2(t-\tau)\,d\tau$
5. $\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\cdot\frac{1}{s}\right\} = \int_0^t \cos 2\tau \cdot 1\,d\tau = \frac{1}{2}\sin 2t$
6. $\mathcal{L}^{-1}\left\{\frac{1}{s-2}\cdot\frac{1}{s+3}\right\} = \int_0^t e^{2\tau}e^{-3(t-\tau)}\,d\tau = \frac{e^{2t}-e^{-3t}}{5}$
7. $\mathcal{L}^{-1}\left\{\frac{2}{s^3}\cdot\frac{1}{s}\right\} = \int_0^t \tau^2\cdot 1\,d\tau = \frac{t^3}{3}$
8. $\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\cdot\frac{3}{s^2+9}\right\} = \frac{1}{3}\int_0^t \sin 3\tau\sin 3(t-\tau)\,d\tau$
9. $\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{s}{s^2-1}\right\} = \int_0^t 1\cdot\cosh(t-\tau)\,d\tau = \sinh t$

Division by s and Partial Fractions

$$\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{F(s)}{s}$$

Partial Fractions: Decompose $F(s)$ into simpler fractions to find $\mathcal{L}^{-1}$.

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Examples:

1. $\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \int_0^t e^{-u}\,du = 1 - e^{-t}$
2. $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \frac{1}{s} - \frac{s}{s^2+1} = 1 - \cos t$
3. $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s-2)}\right\} = \frac{1}{s-2} - \frac{1}{s-1} = e^{2t} - e^{t}$
4. $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)(s+1)}\right\} = \frac{1/5}{s+1} + \frac{4/5s}{s^2+4} = \frac{1}{5}e^{-t} + \frac{4}{5}\cos 2t$
5. $\mathcal{L}^{-1}\left\{\frac{1}{s(s-3)}\right\} = \frac{1}{3}\left(\frac{1}{s} - \frac{1}{s-3}\right) = \frac{1}{3}(1 - e^{3t})$
6. $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+2)}\right\} = \frac{1/4}{s} - \frac{1/4}{s^2} - \frac{1/2}{s+2} = \frac{1}{4} - \frac{t}{4} - \frac{1}{2}e^{-2t}$
7. $\mathcal{L}^{-1}\left\{\frac{s}{(s-1)^2}\right\} = e^{t} + te^{t}$
8. $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+9)}\right\} = \frac{1}{10}e^{-t} - \frac{1}{30}\sin 3t - \frac{1}{10}\cos 3t$
9. $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2-4)}\right\} = \frac{1}{4}\sinh 2t$

Solving Differential Equations with Laplace Transform

Apply $\mathcal{L}$ to transform a differential equation into an algebraic equation, solve for $Y(s)$, and find $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

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Apply $\mathcal{L}$ to transform a differential equation into an algebraic equation, solve for $Y(s)$, and find $y(t) = \mathcal{L}^{-1}\{Y(s)\}$. **Examples**:

1. Solve $y' + y = 1$, $y(0) = 0$.

$$sY(s) - 0 + Y(s) = \frac{1}{s} \implies Y(s) = \frac{1}{s(s+1)} \implies y(t) = 1 - e^{-t}$$

2. Solve $y'' + 4y = 0$, $y(0) = 1$, $y'(0) = 0$.

$$s^2 Y(s) - s + 4Y(s) = 0 \implies Y(s) = \frac{s}{s^2+4} \implies y(t) = \cos 2t$$

3. Solve $y' - 2y = e^{3t}$, $y(0) = 0$.

$$sY(s) - 2Y(s) = \frac{1}{s-3} \implies y(t) = -\frac{1}{5}e^{3t} + \frac{1}{5}e^{2t}$$

4. Solve $y'' + y = \sin t$, $y(0) = 0$, $y'(0) = 0$.

$$Y(s) = \frac{1}{(s^2+1)^2} \implies y(t) = \frac{1}{2}(\sin t - t\cos t)$$

5. Solve $y'' - y = 0$, $y(0) = 1$, $y'(0) = 1$.

$$Y(s) = \frac{s+1}{} \implies y(t) = \cosh t + \sinh t = e^t$$

Fourier Transform: Definition

The **Fourier transform** of $f(t)$ is:

$$\mathcal{F}\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$$

Fourier Sine Transform:

$$F_s(\omega) = \int_0^{\infty} f(t)\sin(\omega t)\,dt$$

Fourier Cosine Transform:

$$F_c(\omega) = \int_0^{\infty} f(t)\cos(\omega t)\,dt$$

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Examples:

1. $\mathcal{F}\{e^{-a|t|}\} = \frac{2a}{a^2+\omega^2}$, $a > 0$
2. $\mathcal{F}\{u(t) - u(t-1)\} = \frac{1-e^{-i\omega}}{i\omega}$
3. $F_s\{e^{-at}\} = \frac{\omega}{a^2+\omega^2}$, $a > 0$
4. $F_c\{e^{-at}\} = \frac{a}{a^2+\omega^2}$, $a > 0$
5. $\mathcal{F}\{e^{-t^2}\} = \sqrt{\pi} e^{-\omega^2/4}$
6. $F_s\{te^{-t}\} = \frac{2\omega}{(1+\omega^2)^2}$

Inverse Fourier Transform

$$\mathcal{F}^{-1}\{F(\omega)\} = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\, d\omega$$

Inverse Sine Transform:

$$f(t) = \frac{2}{\pi}\int_{0}^{\infty} F_s(\omega)\sin(\omega t)\, d\omega$$

Inverse Cosine Transform:

$$f(t) = \frac{2}{\pi}\int_{0}^{\infty} F_c(\omega)\cos(\omega t)\, d\omega$$

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$$f(t) = \frac{2}{\pi}\int_{0}^{\infty} F_c(\omega)\cos(\omega t)\,d\omega$$

Examples:

1. $\mathcal{F}^{-1}\left\{\frac{2a}{a^2+\omega^2}\right\} = e^{-a|t|}$
2. $F_s^{-1}\left\{\frac{\omega}{a^2+\omega^2}\right\} = e^{-at},\ t > 0$
3. $F_c^{-1}\left\{\frac{a}{a^2+\omega^2}\right\} = e^{-at},\ t > 0$
4. $\mathcal{F}^{-1}\left\{\sqrt{\pi}e^{-\omega^2/4}\right\} = e^{-t^2}$
5. $F_s^{-1}\left\{\frac{1}{\omega}\right\} = 1$
6. $F_c^{-1}\left\{-\frac{1}{\omega^2}\right\} = t$

Fourier-Laplace Relation and Properties

Relation: If $f(t) = 0$ for $t < 0$, the Fourier transform of $f(t)$ is the Laplace transform with $s = i\omega$.

$$\mathcal{F}\{f(t)\} = \mathcal{L}\{f(t)\}|_{s=i\omega}$$

Change of Scale:

$$\mathcal{F}\{f(at)\} = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

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Examples (Change of Scale):

1. $\mathcal{F}\{\sin 2t\} = \pi(\delta(\omega - 2) - \delta(\omega + 2))$
2. $\mathcal{F}\{\cos 3t\} = \pi(\delta(\omega - 3) + \delta(\omega + 3))$
3. $\mathcal{F}\{e^{-2t^2}\} = \frac{\sqrt{\pi}}{2} e^{-\omega^2/8}$
4. $\mathcal{F}\{te^{-3t}\} = \frac{1}{(3+i\omega)^2}$, $t > 0$
5. $\mathcal{F}\{\sin(4t)\} = \pi(\delta(\omega - 4) - \delta(\omega + 4))$
6. $F_s\{\sin 2t\} = \frac{2}{4+\omega^2}$
7. $F_c\{\cos 3t\} = \frac{\omega^2}{\omega^2+9}$
8. $\mathcal{F}\{e^{-t/2}\} = \frac{1}{1/2+i\omega}$, $t > 0$
9. $F_s\{te^{-2t}\} = \frac{4\omega}{(4+\omega^2)^2}$
10. $F_c\{t^2 e^{-t}\} = \frac{2(1-3\omega^2)}{(1+\omega^2)^3}$

Modulation, Derivative, and Convolution Theorems

Modulation Theorem:

$$\mathcal{F}\{f(t)\cos\omega_0 t\} = \frac{1}{2}[F(\omega-\omega_0)+F(\omega+\omega_0)]$$

Derivative Theorem:

$$\mathcal{F}\{f'(t)\} = i\omega F(\omega)$$

Convolution Theorem:

$$\mathcal{F}\{f(t)*g(t)\} = F(\omega)G(\omega)$$

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$$\mathcal{F}\{f(t) * g(t)\} = F(\omega)G(\omega)$$

Examples:

1. $\mathcal{F}\{e^{-|t|}\cos 2t\} = \frac{1}{2}\left(\frac{2}{4+(\omega-2)^2} + \frac{2}{4+(\omega+2)^2}\right)$
2. $\mathcal{F}\{te^{-t}\} = \frac{i\omega}{(1+i\omega)^2}$, $t > 0$
3. $\mathcal{F}\{e^{-t} * e^{-2t}\} = \frac{1}{1+i\omega} \cdot \frac{1}{2+i\omega}$, $t > 0$
4. $F_s\{\sin t \cos t\} = \frac{\omega}{2(1+\omega^2)}$
5. $F_c\{e^{-t}\cos 2t\} = \frac{1}{2}\left(\frac{1}{1+(\omega-2)^2} + \frac{1}{1+(\omega+2)^2}\right)$
6. $\mathcal{F}\{t'\} = i\omega \cdot \frac{1}{i\omega} = 1$
7. $\mathcal{F}\{\sin t * \cos t\} = \frac{\pi}{2}(\delta(\omega-1) - \delta(\omega+1)) \cdot \frac{\pi}{2}(\delta(\omega-1) + \delta(\omega+1))$
8. $\mathcal{F}\{(e^{-t})'\} = i\omega \cdot \frac{1}{1+i\omega}$, $t > 0$

Finite Fourier Sine and Cosine Transforms

Finite Fourier Sine Transform over $[0, L]$:

$$F_s(n) = \int_0^L f(t) \sin\left(\frac{n\pi t}{L}\right) dt$$

Finite Fourier Cosine Transform over $[0, L]$:

$$F_c(n) = \int_0^L f(t) \cos\left(\frac{n\pi t}{L}\right) dt$$

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$$F_c(n) = \int_0^L f(t) \cos\left(\frac{n\pi t}{L}\right) dt$$

Examples (over $[0, \pi]$):

1. $F_s\{t\} = \frac{\pi(-1)^{n+1}}{n}$
2. $F_c\{t\} = \frac{\pi[(-1)^n - 1]}{n^2}$, $n \neq 0$; $F_c(0) = \frac{\pi^2}{2}$
3. $F_s\{t^2\} = \frac{2\pi^2(-1)^{n+1}}{n^3} - \frac{2\pi(-1)^n}{n}$
4. $F_c\{t^2\} = \frac{\pi^2(-1)^n}{3}$, $n \neq 0$
5. $F_s\{\sin t\} = \frac{\pi}{2}\delta_{n1}$
6. $F_c\{\cos t\} = \frac{\pi}{2}\delta_{n1}$
7. $F_s\{e^{-t}\} = \frac{n\pi(1+(-1)^{n+1})}{1+n^2\pi^2}$
8. $F_c\{e^{-t}\} = \frac{1-(-1)^n e^{-\pi}}{1+n^2\pi^2}$

Finite Inverse Fourier Transforms

Inverse Sine Transform over $[0, L]$:

$$f(t) = \frac{2}{L} \sum_{n=1}^{\infty} F_s(n) \sin\left(\frac{n\pi t}{L}\right)$$

Inverse Cosine Transform over $[0, L]$:

$$f(t) = \frac{1}{L} F_c(0) + \frac{2}{L} \sum_{n=1}^{\infty} F_c(n) \cos\left(\frac{n\pi t}{L}\right)$$

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Examples (over $[0, \pi]$):

1. $F_s(n) = \frac{\pi(-1)^{n+1}}{n} \implies f(t) = t$
2. $F_c(n) = \frac{\pi[(-1)^n - 1]}{n^2} \implies f(t) = t, n \neq 0$
3. $F_s(n) = \frac{\pi}{2}\delta_{n1} \implies f(t) = \sin t$
4. $F_c(n) = \frac{\pi}{2}\delta_{n1} \implies f(t) = \cos t$
5. $F_s(n) = \frac{2\pi^2(-1)^{n+1}}{n^3} \implies f(t) \approx t^2$
6. $F_c(n) = \frac{\pi^2(-1)^n}{3} \implies f(t) \approx t^2, n \neq 0$
7. $F_s(n) = \frac{n\pi(1+(-1)^{n+1})}{1+n^2\pi^2} \implies f(t) = e^{-t}$
8. $F_c(n) = \frac{1-(-1)^n e^{-\pi}}{1+n^2\pi^2} \implies f(t) = e^{-t}$

Fourier Integral Theorem

Fourier Integral Theorem:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) e^{i\omega(t-u)}\, du\, d\omega$$

Fourier Sine Integral:

$$f(t) = \frac{2}{\pi} \int_{0}^{\infty} \left(\int_{0}^{\infty} f(u) \sin(\omega u)\, du \right) \sin(\omega t)\, d\omega$$

Fourier Cosine Integral:

$$f(t) = \frac{2}{\pi} \int_{0}^{\infty} \left(\int_{0}^{\infty} f(u) \cos(\omega u)\, du \right) \cos(\omega t)\, d\omega$$

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Fourier Cosine Integral:

$$f(t) = \frac{2}{\pi}\int_{0}^{\infty}\left(\int_{0}^{\infty} f(u)\cos(\omega u)\,du\right)\cos(\omega t)\,d\omega$$

Examples:

1. $f(t) = e^{-|t|} \implies \int_0^\infty \frac{2}{1+\omega^2}\cos(\omega t)\,d\omega = \pi e^{-|t|}$
2. $f(t) = e^{-at},\, t > 0 \implies \frac{2}{\pi}\int_0^\infty \frac{a}{a^2+\omega^2}\cos(\omega t)\,d\omega = e^{-at}$
3. $f(t) = te^{-t},\, t > 0 \implies \frac{2}{\pi}\int_0^\infty \frac{2\omega}{(1+\omega^2)^2}\sin(\omega t)\,d\omega = te^{-t}$
4. $f(t) = 1,\, 0 < t < 1 \implies \frac{2}{\pi}\int_0^\infty \frac{\sin\omega}{\omega}\sin(\omega t)\,d\omega$
5. $f(t) = e^{-t^2} \implies \int_0^\infty \sqrt{\pi}e^{-\omega^2/4}\cos(\omega t)\,d\omega = \pi e^{-t^2}$
6. $f(t) = t,\, 0 < t < 1 \implies \frac{2}{\pi}\int_0^\infty \frac{1-\cos\omega}{\omega^2}\cos(\omega t)\,d\omega$

Conclusion

- Laplace transforms convert differential equations into algebraic equations, with properties like shifting and convolution aiding computation.
- Inverse Laplace transforms recover functions using partial fractions and theorems.
- Fourier transforms analyze functions over the entire real line, with sine and cosine variants for specific cases.
- Finite Fourier transforms and integrals provide tools for bounded intervals, with applications in signal processing and differential equations.