# Fourier Series

## Periodic Functions, Expansions, and Half-Range Series

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# Outline

# Periodic Functions

A function $f(x)$ is **periodic** with period $T$ if:

$$f(x+T) = f(x) \quad \text{for all } x$$

- Common periods: $2\pi$, $2c$, etc.
- Examples: $\sin x$, $\cos x$ (period $2\pi$), $\sin(nx)$, $\cos(nx)$.

# Even and Odd Functions

- **Even function**: $f(-x) = f(x)$
  - Example: $\cos x$, $x^2$
  - Fourier series contains only cosine terms.
- **Odd function**: $f(-x) = -f(x)$
  - Example: $\sin x$, $x$, $x^3$
  - Fourier series contains only sine terms.

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**Examples**:

1. $f(x) = x^2$ is even: $f(-x) = (-x)^2 = x^2$.
2. $f(x) = x^3$ is odd: $f(-x) = (-x)^3 = -x^3$.
3. $f(x) = \cos x$ is even.
4. $f(x) = \sin x$ is odd.
5. $f(x) = x \sin x$ is even: $f(-x) = (-x)\sin(-x) = x \sin x$.
6. $f(x) = x \cos x$ is odd.
7. $f(x) = e^x$ is neither even nor odd.
8. $f(x) = |x|$ is even.
9. $f(x) = x^4$ is even.
10. $f(x) = \tan x$ is odd.

# Fourier Series Expansion

For a periodic function $f(x)$ with period $2L$, the Fourier series is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

where:

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx, \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx, \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}$$

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Intervals: $[-\pi, \pi]$ $(L = \pi)$, $[0, 2\pi]$, $[-c, c]$ $(L = c)$, $[0, 2c]$ $(L = c)$.

# Fourier Series over $[-\pi, \pi]$

**Examples**:

1. $f(x) = x$, period $2\pi$.

$$a_0 = 0, \quad a_n = 0, \quad b_n = \frac{2(-1)^{n+1}}{n}, \quad f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

2. $f(x) = x^2$.

$$a_0 = \frac{2\pi^2}{3}, \quad a_n = \frac{4(-1)^n}{n^2}, \quad b_n = 0, \quad f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

3. $f(x) = |x|$.

$$a_0 = \pi, \quad a_n = \frac{2(-1)^{n+1} + 2}{\pi n^2}, \quad b_n = 0, \quad f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$$

# Fourier Series over $[0, 2\pi]$

**Examples**:

4. $f(x) = x$.

$$a_0 = \pi, \quad a_n = 0, \quad b_n = -\frac{2}{n}, \quad f(x) = \pi - 2\sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

5. $f(x) = x^2$.

$$a_0 = \frac{4\pi^2}{3}, \quad a_n = \frac{4}{n^2}, \quad b_n = -\frac{4\pi}{n}, \quad f(x) = \frac{2\pi^2}{3} + 4\sum_{n=1}^{\infty}\left(\frac{\cos nx}{n^2} - \frac{\pi \sin}{n}\right.$$

6. $f(x) = e^x$.

$$a_0 = \frac{e^{2\pi} - 1}{\pi}, \quad a_n = \frac{e^{2\pi} - (-1)^n}{\pi(1+n^2)}, \quad b_n = \frac{n(e^{2\pi} + (-1)^n)}{\pi(1+n^2)}$$

# Fourier Series over $[-c, c]$

**Examples**:

7. $f(x) = x$, $L = c$.

$$a_n = 0, \quad b_n = \frac{2c(-1)^{n+1}}{n\pi}, \quad f(x) = 2c \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin \frac{n\pi x}{c}$$

8. $f(x) = x^2$, $L = c$.

$$a_0 = \frac{2c^2}{3}, \quad a_n = \frac{4c^2(-1)^n}{n^2\pi^2}, \quad b_n = 0, \quad f(x) = \frac{c^2}{3} + \frac{4c^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos$$

9. $f(x) = |x|$, $L = c$.

$$a_0 = c, \quad a_n = \frac{2c[(-1)^n - 1]}{n^2\pi^2}, \quad b_n = 0, \quad f(x) = \frac{c}{2} - \frac{2c}{\pi^2} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2}$$

# Fourier Series over $[0, 2c]$

**Examples**:

10. $f(x) = x$, $L = c$.

$$a_0 = c, \quad a_n = 0, \quad b_n = -\frac{2c}{n\pi}, \quad f(x) = \frac{c}{2} - \frac{2c}{\pi}\sum_{n=1}^{\infty} \frac{\sin \frac{n\pi x}{c}}{n}$$

11. $f(x) = x^2$, $L = c$.

$$a_0 = \frac{4c^2}{3}, \quad a_n = \frac{4c^2}{n^2\pi^2}, \quad b_n = -\frac{4c^2}{n\pi}, \quad f(x) = \frac{2c^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4c^2}{n^2\pi^2}\cos\frac{n\pi x}{c}\right.$$

12. $f(x) = \sin\frac{\pi x}{2c}$, $L = c$.

$$a_n = 0, \quad b_1 = 1, \quad b_n = 0 \text{ for } n \neq 1, \quad f(x) = \sin\frac{\pi x}{2c}$$

# Fourier Sine and Cosine Series

For a function $f(x)$ on $[0, L]$:

- **Cosine series** (even extension):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx$$

- **Sine series** (odd extension):

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

# Fourier Cosine Series Examples

**Examples (over $[0, \pi]$):**

1. $f(x) = x$.

$$a_0 = \pi, \quad a_n = \frac{2[(-1)^n - 1]}{n^2\pi}, \quad f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$$

2. $f(x) = x^2$.

$$a_0 = \frac{2\pi^2}{3}, \quad a_n = \frac{4(-1)^n}{n^2}, \quad f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx$$

3. $f(x) = 1$.

$$a_0 = 2, \quad a_n = 0, \quad f(x) = 1$$

4. $f(x) = \sin x$.

$$a_0 = \frac{4}{\pi}, \quad a_n = \frac{2(-1)^n}{\pi(1-n^2)} \text{ for } n \neq 1, \quad a_1 = 0$$

5. $f(x) = x^3$.

# Fourier Cosine Series Examples (Cont.)

6. $f(x) = e^x$, $[0, 1]$.

$$a_0 = 2(e-1), \quad a_n = \frac{2[e(-1)^n - 1]}{1 + n^2\pi^2}$$

7. $f(x) = x^4$, $[0, \pi]$.

$$a_0 = \frac{2\pi^4}{5}, \quad a_n = \frac{8(-1)^n(3\pi^2 - 2n^2\pi^2)}{n^4\pi}$$

8. $f(x) = \cos x$, $[0, \pi]$.

$$a_0 = \frac{2}{\pi}, \quad a_1 = 1, \quad a_n = 0 \text{ for } n \neq 1$$

9. $f(x) = x(1-x)$, $[0, 1]$.

$$a_0 = \frac{2}{3}, \quad a_n = \frac{4[(-1)^n - 1]}{n^2\pi^2}$$

10. $f(x) = x^2(1-x)$, $[0, 1]$.

# Fourier Sine Series Examples

**Examples (over $[0, \pi]$):**

1. $f(x) = x$.

$$b_n = \frac{2(-1)^{n+1}}{n}, \quad f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

2. $f(x) = x^2$.

$$b_n = \frac{2[(-1)^n 2\pi n - \pi(-1)^n + 2]}{n^3\pi}$$

3. $f(x) = \sin x$.

$$b_1 = 1, \quad b_n = 0 \text{ for } n \neq 1, \quad f(x) = \sin x$$

4. $f(x) = x^3$.

$$b_n = \frac{2[(-1)^n(3\pi^2 n^2 - 6) + 6]}{n^3\pi}$$

5. $f(x) = 1$.

$$b_n = \frac{2[1 - (-1)^n]}{}$$

# Fourier Sine Series Examples (Cont.)

6. $f(x) = e^x$, $[0, 1]$.

$$b_n = \frac{2n\pi[e(-1)^n + 1]}{1 + n^2\pi^2}$$

7. $f(x) = x(1 - x)$, $[0, 1]$.

$$b_n = \frac{4[(-1)^n + 1]}{n^3\pi^3}$$

8. $f(x) = \cos x$, $[0, \pi]$.

$$b_n = \frac{2n[1 + (-1)^{n+1}]}{\pi(1 - n^2)} \text{ for } n \neq 1, \quad b_1 = 0$$

9. $f(x) = x^4$, $[0, \pi]$.

$$b_n = \frac{2[(-1)^n(4\pi^3 n^3 - 24\pi n) + 24\pi n]}{n^5\pi}$$

10. $f(x) = x^2(1 - x)$, $[0, 1]$.

$$4[(-1)^n(2 - n^2\pi^2) + n^2\pi^2 + 2]$$

# Half-Range Series Expansion

For a function $f(x)$ defined on $[0, L]$, extend to $[-L, L]$:

- **Even extension**: Leads to cosine series.
- **Odd extension**: Leads to sine series.

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- **Even extension**: Leads to cosine series.
- **Odd extension**: Leads to sine series.

**Examples (Cosine, $[0, \pi]$):**

1. $f(x) = x$.

$$a_0 = \pi, \quad a_n = \frac{2[(-1)^n - 1]}{n^2\pi}, \quad f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$$

2. $f(x) = x^2$.

$$a_0 = \frac{2\pi^2}{3}, \quad a_n = \frac{4(-1)^n}{n^2}$$

3. $f(x) = 1$.

$$a_0 = 2, \quad a_n = 0$$

# Half-Range Cosine Examples (Cont.)

4. $f(x) = x^3$, $[0, \pi]$.

$$a_0 = \frac{\pi^3}{2}, \quad a_n = \frac{2[(-1)^n \pi^3 - 6(-1)^n \pi]}{n^2 \pi}$$

5. $f(x) = \sin x$, $[0, \pi]$.

$$a_0 = \frac{4}{\pi}, \quad a_n = \frac{2(-1)^n}{\pi(1 - n^2)} \text{ for } n \neq 1, \quad a_1 = 0$$

6. $f(x) = e^x$, $[0, 1]$.

$$a_0 = 2(e - 1), \quad a_n = \frac{2[e(-1)^n - 1]}{1 + n^2\pi^2}$$

7. $f(x) = x(1 - x)$, $[0, 1]$.

$$a_0 = \frac{2}{3}, \quad a_n = \frac{4[(-1)^n - 1]}{n^2\pi^2}$$

# Half-Range Cosine Examples (Cont.)

8. $f(x) = x^4$, $[0, \pi]$.

$$a_0 = \frac{2\pi^4}{5}, \quad a_n = \frac{8(-1)^n(3\pi^2 - 2n^2\pi^2)}{n^4\pi}$$

9. $f(x) = \cos x$, $[0, \pi]$.

$$a_0 = \frac{2}{\pi}, \quad a_1 = 1, \quad a_n = 0 \text{ for } n \neq 1$$

10. $f(x) = x^2(1 - x)$, $[0, 1]$.

$$a_0 = \frac{1}{6}, \quad a_n = \frac{2[(-1)^n(2 - n^2\pi^2) + 2]}{n^3\pi^3}$$

# Half-Range Sine Examples

**Examples (Sine, $[0, \pi]$):**

1. $f(x) = x$.

$$b_n = \frac{2(-1)^{n+1}}{n}, \quad f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

2. $f(x) = x^2$.

$$b_n = \frac{2[(-1)^n 2\pi n - \pi(-1)^n + 2]}{n^3\pi}$$

3. $f(x) = \sin x$.

$$b_1 = 1, \quad b_n = 0 \text{ for } n \neq 1$$

4. $f(x) = x^3$.

$$b_n = \frac{2[(-1)^n(3\pi^2 n^2 - 6) + 6]}{n^3\pi}$$

5. $f(x) = 1$.

$$b_n = \frac{2[1 - (-1)^n]}{n\pi}$$

# Half-Range Sine Examples (Cont.)

6. $f(x) = e^x$, $[0, 1]$.

$$b_n = \frac{2n\pi[e(-1)^n + 1]}{1 + n^2\pi^2}$$

7. $f(x) = x(1 - x)$, $[0, 1]$.

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$$b_n = \frac{2n[1 + (-1)^{n+1}]}{\pi(1 - n^2)} \text{ for } n \neq 1, \quad b_1 = 0$$

9. $f(x) = x^4$, $[0, \pi]$.

$$b_n = \frac{2[(-1)^n(4\pi^3 n^3 - 24\pi n) + 24\pi n]}{n^5\pi}$$

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$$4[(-1)^n(2 - n^2\pi^2) + n^2\pi^2 + 2]$$

# Conclusion

- Fourier series represent periodic functions using sines and cosines.
- Even and odd functions simplify series to cosine or sine terms, respectively.
- Expansions are valid over various intervals like $[-\pi, \pi]$, $[0, 2\pi]$, $[-c, c]$, $[0, 2c]$.
- Half-range expansions use even or odd extensions for efficient computation.