Differentiation under Integral Sign and Error Function

Cases, Leibniz's Rule, and Error Function Properties

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Outline

Differentiation under Integral Sign: Constant Limits

For a function $f(x, \alpha)$ with constant limits of integration a and b, the derivative of the integral with respect to a parameter α is:

$$\frac{d}{d\alpha} \int_a^b f(x, \alpha)\, dx = \int_a^b \frac{\partial}{\partial \alpha} f(x, \alpha)\, dx$$

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- Assumes $f(x, \alpha)$ and $\frac{\partial f}{\partial \alpha}$ are continuous.
- Allows exchanging differentiation and integration for constant limits.

Problem with One Parameter

Example

Evaluate $\frac{d}{d\alpha}\int_0^1 e^{\alpha x}\,dx$.

Compute the integral first:

$$I(\alpha) = \int_0^1 e^{\alpha x}\,dx = \left[\frac{1}{\alpha}e^{\alpha x}\right]_0^1 = \frac{e^{\alpha} - 1}{\alpha}$$

Differentiate with respect to α:

$$\frac{d}{d\alpha}I(\alpha) = \frac{d}{d\alpha}\left(\frac{e^{\alpha} - 1}{\alpha}\right) = \frac{\alpha e^{\alpha} - (e^{\alpha} - 1)}{\alpha^2} = \frac{e^{\alpha}(\alpha - 1) + 1}{\alpha^2}$$

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Using differentiation under the integral sign:

$$\frac{d}{d\alpha}\int_0^1 e^{\alpha x}\,dx = \int_0^1 \frac{\partial}{\partial\alpha}e^{\alpha x}\,dx = \int_0^1 xe^{\alpha x}\,dx$$

Compute:

$$\int_0^1 xe^{\alpha x}\,dx = \left[\frac{xe^{\alpha x}}{\alpha}\right]_0^1 - \int_0^1 \frac{e^{\alpha x}}{\alpha}\,dx = \frac{e^{\alpha}}{\alpha} - \frac{e^{\alpha}-1}{\alpha^2} = \frac{e^{\alpha}(\alpha-1)+1}{\alpha^2}$$

Both methods agree

Problem with Two Parameters

Example

Evaluate $\frac{\partial}{\partial \alpha} \int_0^1 \frac{x}{\alpha + \beta x} \, dx$.

Using differentiation under the integral sign:

$$\frac{\partial}{\partial \alpha} \int_0^1 \frac{x}{\alpha + \beta x} \, dx = \int_0^1 \frac{\partial}{\partial \alpha} \left(\frac{x}{\alpha + \beta x} \right) dx = \int_0^1 \frac{-x}{(\alpha + \beta x)^2} \, dx$$

Problem with Two Parameters

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Compute the integral:

$$\int_0^1 \frac{-x}{(\alpha+\beta x)^2}\,dx = -\int_0^1 \frac{x}{(\alpha+\beta x)^2}\,dx$$

Substitute $u = \alpha + \beta x$, $du = \beta dx$, $x = \frac{u-\alpha}{\beta}$, $dx = \frac{du}{\beta}$:

$$-\int_\alpha^{\alpha+\beta} \frac{\frac{u-\alpha}{\beta}}{u^2}\cdot\frac{du}{\beta} = -\frac{1}{\beta^2}\int_\alpha^{\alpha+\beta}\frac{u-\alpha}{u^2}\,du = -\frac{1}{\beta^2}\int_\alpha^{\alpha+\beta}\left(\frac{1}{u}-\frac{\alpha}{u^2}\right)du$$

$$= -\frac{1}{\beta^2}\left[\ln u + \frac{\alpha}{u}\right]_\alpha^{\alpha+\beta} = -\frac{1}{\beta^2}\left(\ln(\alpha+\beta) - \ln\alpha + \frac{\alpha}{\alpha+\beta} - \frac{\alpha}{\alpha}\right) = -\frac{1}{\beta^2}\left(\ln\frac{\alpha+\beta}{\alpha}\right.$$

Leibniz's Rule for Differentiation under Integral Sign

For an integral with variable limits $a(\alpha)$ and $b(\alpha)$:

$$\frac{d}{d\alpha}\int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\,dx = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial f}{\partial \alpha}\,dx + f(b(\alpha),\alpha)\frac{db}{d\alpha} - f(a(\alpha),\alpha)\frac{da}{d\alpha}$$

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- Accounts for the effect of changing limits and the integrand.
- Requires f, $\frac{\partial f}{\partial \alpha}$, $a(\alpha)$, and $b(\alpha)$ to be appropriately continuous and differentiable.

Examples on Leibniz's Rule

Example

Evaluate $\frac{d}{d\alpha}\int_0^\alpha x^2 e^{\alpha x}\,dx$.

Apply Leibniz's rule with $f(x,\alpha) = x^2 e^{\alpha x}$, $a(\alpha) = 0$, $b(\alpha) = \alpha$:

$$\frac{d}{d\alpha}\int_0^\alpha x^2 e^{\alpha x}\,dx = \int_0^\alpha \frac{\partial}{\partial\alpha}(x^2 e^{\alpha x})\,dx + f(\alpha,\alpha)\cdot\frac{d}{d\alpha}(\alpha) - f(0,\alpha)\cdot\frac{d}{d\alpha}(0)$$

$$= \int_0^\alpha x^3 e^{\alpha x}\,dx + \alpha^2 e^{\alpha^2}\cdot 1 - 0$$

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Compute the integral using integration by parts (details omitted for brevity):

$$\int_0^\alpha x^3 e^{\alpha x}\,dx = \left[\frac{x^3 e^{\alpha x}}{\alpha}\right]_0^\alpha - \frac{3}{\alpha}\int_0^\alpha x^2 e^{\alpha x}\,dx$$

This requires further integration, but Leibniz's rule simplifies the process.

Definition of Error Function

The **error function** is defined as:

$$(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt$$

The **complementary error function** is:

$$(z) = 1 - (z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt$$

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- Used in probability, statistics, and diffusion processes.
- (z) is odd: $(-z) = -(z)$.

Properties of Error Function

1. **Special Values**:
   - $(0) = 0$
   - $\lim_{z\to\infty}(z) = 1$, $\lim_{z\to-\infty}(z) = -1$
   - $(\infty) = 0$, $(0) = 1$
2. **Derivative**:

$$\frac{d}{dz}(z) = \frac{2}{\sqrt{\pi}} e^{-z^2}$$

3. **Relation to Gamma Function**:

$$(z) = \frac{1}{\sqrt{\pi}} \gamma\left(\frac{1}{2}, z^2\right), \quad \gamma(a, x) = \int_0^x t^{a-1} e^{-t}\, dt$$

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Proof of Derivative:

$$\frac{d}{dz}(z) = \frac{d}{dz}\left(\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\, dt\right) = \frac{2}{\sqrt{\pi}} e^{-z^2}$$

By the fundamental theorem of calculus, since the integrand is continuous.

Conclusion

- Differentiation under the integral sign simplifies parameter-dependent integrals.
- Leibniz's rule extends this to variable limits, incorporating boundary terms.
- The error function and its complement are key in probability and have well-defined properties.
- Examples demonstrate practical applications of these techniques.