Beta and Gamma Functions

Definitions, Properties, Examples, and Relations

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Outline

Definition of Gamma Function

The Gamma function, denoted $\Gamma(z)$, is defined for $\Re(z) > 0$ as:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$$

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$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt$$

- It generalizes the factorial for non-integer values.
- For positive integers n, $\Gamma(n) = (n-1)!$.

Properties of Gamma Function

1. **Recurrence Relation**: $\Gamma(z+1) = z\Gamma(z)$
2. **Special Values**:
   - $\Gamma(1) = 1$
   - $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
3. **Reflection Formula**: $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$

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Proof of Recurrence Relation:

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t}\, dt$$

Using integration by parts ($u = t^z$, $dv = e^{-t}dt$):

$$\Gamma(z+1) = \left[-t^z e^{-t}\right]_0^\infty + \int_0^\infty z t^{z-1} e^{-t}\, dt = 0 + z\Gamma(z)$$

Examples on Gamma Function

Example

Compute $\Gamma(3)$.

Using the recurrence relation:

$$\Gamma(3) = \Gamma(2+1) = 2\Gamma(2) = 2 \cdot 1 \cdot \Gamma(1) = 2 \cdot 1 \cdot 1 = 2$$

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Example

Compute $\Gamma\left(\frac{3}{2}\right)$.

Using $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$:

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(1 + \frac{1}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

Definition of Beta Function

The Beta function, denoted $B(m,n)$, is defined for $\Re(m) > 0$, $\Re(n) > 0$ as:

$$B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,dt$$

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The Beta function, denoted $B(m, n)$, is defined for $\Re(m) > 0$, $\Re(n) > 0$ as:

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- Alternatively, it can be expressed as:

$$B(m,n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\,dt$$

- Symmetric: $B(m, n) = B(n, m)$.

Properties of Beta Function

1. **Symmetry**: $B(m, n) = B(n, m)$
2. **Relation to Gamma Function** (proved later):

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

3. **Special Values**:
   - $B(1, 1) = 1$
   - $B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$

Examples on Beta Function

Example

Compute $B(2,3)$.

Using the integral definition:

$$B(2,3) = \int_0^1 t^{2-1}(1-t)^{3-1}\,dt = \int_0^1 t(1-t)^2\,dt$$

Let $I = \int_0^1 t(1-t)^2\,dt$. Expand and integrate:

$$I = \int_0^1 (t-2t^2+t^3)\,dt = \left[\frac{t^2}{2} - \frac{2t^3}{3} + \frac{t^4}{4}\right]_0^1 = \frac{1}{2}-\frac{2}{3}+\frac{1}{4} = \frac{6-8+3}{12} = \frac{1}{12}$$

Examples on Beta Function

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Example

Compute $B\left(1, \frac{1}{2}\right)$.

Relation between Beta and Gamma Function

Theorem: For $\Re(m) > 0$, $\Re(n) > 0$,

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

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Proof: Start with the product of two Gamma functions:

$$\Gamma(m)\Gamma(n) = \left(\int_0^\infty t^{m-1} e^{-t}\, dt\right) \left(\int_0^\infty s^{n-1} e^{-s}\, ds\right)$$

Combine into a double integral:

$$\Gamma(m)\Gamma(n) = \int_0^\infty \int_0^\infty t^{m-1} s^{n-1} e^{-(t+s)}\, dt\, ds$$

Substitute $t = uv$, $s = u(1-v)$, with Jacobian u:

$$\Gamma(m)\Gamma(n) = \int_0^\infty \int_0^1 (uv)^{m-1} [u(1-v)]^{n-1} e^{-u} u\, dv\, du$$

Example Using Beta-Gamma Relation

Example

Verify $B(2,3)$ using the Beta-Gamma relation.

From the relation:

$$B(2,3) = \frac{\Gamma(2)\Gamma(3)}{\Gamma(2+3)} = \frac{\Gamma(2)\Gamma(3)}{\Gamma(5)}$$

Compute:

$$\Gamma(2) = 1, \quad \Gamma(3) = 2, \quad \Gamma(5) = 4! = 24$$

$$B(2,3) = \frac{1 \cdot 2}{24} = \frac{2}{24} = \frac{1}{12}$$

This matches the earlier result from the integral definition.

Conclusion

- The Gamma function extends the factorial to real and complex numbers.
- The Beta function is closely related to integrals over $[0, 1]$ and is symmetric.
- The relation $B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ connects the two functions.
- Examples and proofs illustrate their properties and applications.