Date:-
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Lattice Theory

Ordered Relation or Partially ordering ($\leq$):-

$a \leq b$:) a related to b

i) Reflexivity :-

$\forall a, \quad a \leq a$

ii) Anti-Symmetric :-

$a \leq b$ and $b \leq a \Leftrightarrow a = b$

iii) Transitivity :-

$a \leq b$ and $b \leq c \Rightarrow a \leq c$

iv) Linearity :-

Either $a \leq b$ or $b \leq a \quad \forall a, b$

Then the relation '$\leq$' is called ordered relation or partially ordering.

Equivalence relation:-

If a non-empty set P or $\phi \neq P$ is a set with ordered relation ($\leq$) is called equivalence relation, if,

i) Reflexitivity :-

$a \leq a \quad \forall a \in P$

ii) Symmetry :-

If $a \leq b \Rightarrow b \leq a$, $a, b \in P$

iii) Transitivity :-

If $a \leq b$ and $b \leq c \Rightarrow a \leq c$, $a, b, c \in P$

Partial order Relation:-

If a non-empty set P or $\phi \neq P$ is a set with ordered relation is called Partial ordered relation if,

i) Reflexivity :-

$a \leq a \quad \forall a \in P$

ii) Anti - symmetric :-
If $a \le b$ and $b \le a \Rightarrow a = b$ $\forall a, b \in P$

iii) Transitivity :-
If $a \le b$ and $b \le c \Rightarrow a \le c$ $\forall a, b, c \in P$

Note:-
1) The relation $(\le)$ is not necessary ordered relation in above two definitions.

Examples:-

Sr.No.	Relation on set (Relation)	Equivalence Relation	Partial order Relation
1.	$(\mathbb{N}, \le)$	No. (symmetry fails) $2 \le 3 \not\Rightarrow 3 \le 2$	YES $1 \le 1, 1 \le 1 \Rightarrow 1 = 1$ $2 \le 2, 2 \le 2 \Rightarrow 2 = 2$
2.	Let, $\phi \ne A$ be a set $(P(A), \le)$ $\le$ = Set of inclusion $(\subseteq)$	No. (symmetry fails) $\{1\} \subseteq \{1,2\} \not\Rightarrow \{1,2\} \subseteq \{1\}$	YES $\{1\} \subseteq \{1\}$
3.	$A = \{1, 2\}$ $R(\le) = \{(1,1)(2,2)\} \subseteq A \times A$ $(A, \le)$	YES	YES
4.	$A = \{1, 2\}$ $R(\le) = \{(1,1)(1,2)(2,1)(2,2)\} \subseteq A \times A$ i.e. $(A, \le)$	YES	No. (Bcoz, $1 \le 2$ & $2 \le 1 \not\Rightarrow 1 = 2$ (Anti-Symmetry fails).

Q. Given an relation which is neither equivalence nor partial ordered.

→ Let,

consider the set of integers $\mathbb{Z}$.

$R(\le)$ be the relation on $\mathbb{Z}$.

$R(\le) = \{(a,b) : ab \ge 0\}$

i) Reflexive: $a \le a \Rightarrow a \cdot a \ge 0$

ii) Symmetric: $a \le b \Rightarrow b \le a$ i.e. $ab \ge 0 \Rightarrow ba \ge 0$

iii) Transitivity: [illegible]

[illegible] $ab \ge 0$, $bc \ge 0 \Rightarrow ac \ge 0$

[illegible]

[illegible] $bc \ge 0$ but $ac < 0$

∴ Transitivity fails

∴ ($\mathbb{Z}$, $ab \ge 0$) is not equivalence relation.

It's Partial ordered set [illegible] symmetry fails [illegible]

i.e. [illegible] neither equivalence nor partially ordered relation.

Partially ordered set:- (POSET)

A set $\phi \ne P$ with relation $(\le)$ is called partially ordered set (POSET) if the relation $(\le)$ is partially ordered.

e.g.

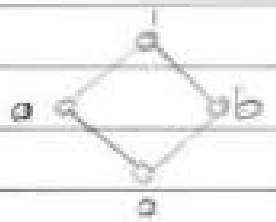
$(\mathbb{N}, \le)$ is partially ordered relation.

Totally ordered Set :- (TOSET)

A set which is non-empty or a set $\phi \neq P$ with relation is called TOSET or totally ordered set if the relation is ordered relation.

Note:-

1] Every TOSET is POSET but converse need not be true.

1] 

Po-set but not To-set , To-set $\Rightarrow$ Po-set

Note:-

1] Notation of Po-set

Let $\phi \neq P$ be a set and $\leq$ - Partial ordered relation on P.

Then we denote it by,

$(P \ \leq)$ OR $\langle P \ \leq \rangle$ is Po-set.

2] $\langle P \ \leq \rangle$ is simply P - poset.

3] $\langle P \ \leq \rangle \cong \{ (a, b) \in P \times P \mid a \leq b \}$

1] Let $P = \{1, 3, 5, 10, 30\}$ and we define $a \leq b$ iff $a | b$ $\forall$ $a, b \in P$, then

$\langle P \ \leq \rangle = \{ (1 \ 1) (3 \ 3) (5 \ 5) (10 \ 10) (30 \ 30)$
$(1 \ 3) (1 \ 5) (1 \ 10) (1 \ 30) (3 \ 30)$
$(5 \ 10) (5 \ 30) (10 \ 30) \}$.

Clearly, $\langle P \leq \rangle$ is PO-set.

Comparable Elements:-

Let $\langle P \leq \rangle$ is PO-set. Let, $a, b \in P$ be any elements, then a & b are called comparable elements if,

either $a \leq b$ or $b \leq a$.

e.g.

If $P = \{1, 3, 5, 10, 30\}$ and we define $a \leq b$ iff $a|b$ $\forall a, b \in P$. Then, $\langle P \leq \rangle$ is PO-set.

i) 1 and 3

As $1/3$

$\Leftrightarrow 1 \leq 3$

$\therefore$ 1 and 3 are comparable elements.

ii) 3 and 5

As $3 \nmid 5$ and $5 \nmid 3$

$\Rightarrow 3 \nleq 5$ and $5 \nleq 3$

$\therefore$ 3 and 5 are not comparable.

iii) 30 and 5

As $5/30$

$\Leftrightarrow 5 \leq 30$

$\therefore$ 30 and 5 are comparable.

Parallel Elements:- (II)

Let $\langle P \leq \rangle$ is PO-set. Let $a, b \in P$ be any elements then a & b are called parallel elements if,

$a \nleq b$ and $b \nleq a$.

OR

The elements which are not-comparable are called parallel elements.

e.g.
If $P = \{1, 3, 5, 10, 30\}$ and we define $a \le b$ iff $a | b$ $\forall a, b \in P$ then $\langle P, \le \rangle$ is po-set.

i) 1 and 3

As $1 / 3$

$\Leftrightarrow 1 \le 3$

$\therefore$ 1 and 3 are not parallel.

ii) 3 and 5

As $3 \nmid 5$ and $5 \nmid 3$

$\Rightarrow 3 \nleq 5$ and $5 \nleq 3$

$\therefore$ 3 and 5 are parallel.

iii) 30 and 5

As $5 / 30$

$\Rightarrow 5 \le 30$

$\therefore$ 30 and 5 are not parallel.

iv) 3 and 10

As $3 \nmid 10$ and $10 \nmid 3$

$\Rightarrow 3 \nleq 10$ and $10 \nleq 3$

$\therefore$ 3 and 10 are parallel.

v) 30 and 10

As ~~$30 \nmid 10$~~ $10 / 30$

$\Rightarrow 10 \le 30$

$\therefore$ 10 and 30 are comparable, i.e. not parallel.

Chain :-

A PO-set is called chain if any two elements ~~of PO-set~~ are comparable.

e.g.

1) If $P = \{1, 3, 5, 10, 30\}$ and we define $a \le b$ iff $a \mid b$ $\forall\, a, b \in P$, then $\langle P\ \le \rangle$ is PO-set.

But, $\langle P\ \le \rangle$ is not chain.

Because,

$\exists$ 3 and 5 or $3, 5 \in P$ such that $3 \parallel 5$. (3 parallel to 5).

2) If $Q = \{1, 5, 10, 30\}$ and we define $a \le b$ iff $a \mid b$ $\forall\, a, b \in Q$, then $\langle Q\ \le \rangle$ is PO-set.

Clearly, $\langle Q\ \le \rangle$ is chain.

Because,

$\forall\, a, b \in Q$ either $a \le b$ or $b \le a$.

Anti-chain :-

A Po-set P is anti-chain if no two elements of P are comparable.

i.e. $\forall\, a, b \in P \Rightarrow a \parallel b$.

e.g.

If $P = \{1, 2, 3, 4, 5, 6, 9, 12, 18\}$ is Po-set under $a \le b$ iff $a \mid b$.

$\{1, 2\}$	$\{3, 5\}$	$\{1, 3, 2\}$
$\{1, 2, 4\}$	$\{2, 5\}$	$\{1, 4, 9\}$
$\{1, 2, 4, 12\}$	$\{4, 6\}$	P

Chain	Anti-chain	Not chain/ Not Antichain
$\{2, 6, 12\}$	$\{4, 6, 9\}$	
$\{6, 18\}$	$\{2, 5, 9\}$	
$\{1, 3, 9, 18\}$	$\{3, 5, 2\}$	
	$\{4, 5, 6, 9\}$	

Length of Chain:-

Number of elements in chain minus 1 (No. of elements in chain -1) is called length of chain.

If C is a chain then, $l(C) = n(C) - 1$.

Length of Anti-chain:-

Number of elements in anti-chain is called length of anti-chain.

Maximal Chain:-

A chain is maximal if no other element from Poset is comparable with this chain.

OR

A chain is maximal if it not contained properly in another chain

Maximal Antichain:-

An antichain is maximal if no other element from Poset is comparable with this antichain.

OR

A antichain is maximal if it not contained properly in another antichain.

Maximum chain / logest or largest chain:-

The longest among all chains is called maximum chain.

Maximum anti-chain / largest anti-chain:-

The longest among all anti-chains is called maximum anti-chain.

1] If $P = \{1, 2, 3, 4, 5, 6, 9, 12, 18\}$ is a poset under $a \leq b$ iff $a | b$.

Chain		Maximal	Maximum
$\{1, 2\}$	$\subseteq \{1, 2, 4\}$	No	No
$\{1, 2, 4\}$	$\subseteq \{1, 2, 4, 12\}$	No	No
$\{1, 2, 4, 12\}$		YES	YES
$\{3, 6, 12\}$		No	No
$\{6, 18\}$		No	No
$\{1, 3, 9, 18\}$		YES	YES
$\{1, 5\}$		YES	No

Note:-

1] Every maximum chain is maximal but converse may or may not be true.

2] If $P = \{1, 2, 3, 4, 5, 6, 9, 12, 18\}$ is a poset under $a \leq b$ iff $a | b$.

Anti-chain	Maximal	Maximum
$\{3, 5\}$	No	No
$\{2, 5\}$	No	No
$\{4, 6\}$	No	No

Chain	Maximal	Maximum
{4, 6, 9}	No	No
{3, 5, 2}	YES	No
{2, 5, 9}	YES	No
{4, 5, 6, 9}	YES	YES

Note:-

1) Every maximum anti chain is maximal but converse may or may not be true.

Length of Poset:-

A Poset (P ≤) containing finite chain. Let c be a finite longest or maximum chain in P then length of c is called length of Po-set.

Width of Poset:-

A Po-set (P ≤) containing finite antichain. Let c be a finite longest or maximum antichain in P then length of c is called width of PO-set.

OR

'n' is width of Po-set P iff there is anti-chain in P of length 'n' and all anti-chains in P has less or equal to n-elements.

1) Let $P = \{1, 2, 3, 4, 6, 9\}$ then $\langle P \mid \rangle$ is Poset i.e. $a \leq b$ iff $a|b$.

i) Give some antichains and it's lengths.

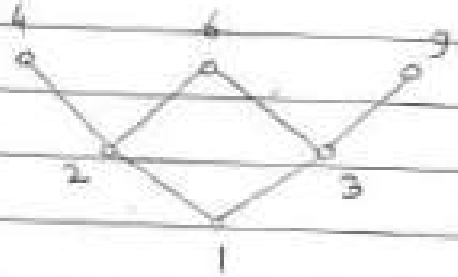
ii) Give some chains and it's lengths.

iii) Find length and width of poset.

Let,

Given that,

$P = \{1, 2, 3, 4, 6, 9\}$ is poset under $a \leq b$ iff $a | b$.



Chains	Length	Anti-chain	Length
$\{1, 2, 4\}$	2	$\{2, 3\}$	2
$\{1, 3, 9\}$	2	$\{4, 6, 9\}$	3
$\{1, 3, 6\}$	2	$\{2, 9\}$	2
$\{1, 2, 6\}$	2	$\{3, 4\}$	2
$\{1, 2\}$	1	$\{6, 9\}$	2
$\{3, 9\}$	1	$\{4, 9\}$	2
$\{3, 6\}$	1		
$\{1, 9\}$	1		

3] By definition, length of poset = length of maximum / longest chain in P

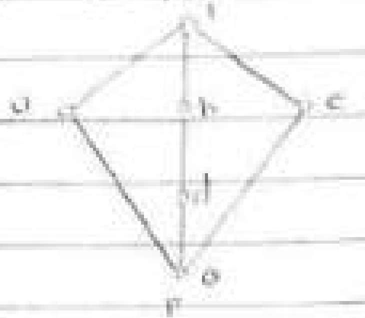
$= 2$

Similarly, by def^n,

Width of poset = length of logest antichain in P

$= 3$

2] Consider a Poset P defined by.



i] Give some anti-chains and it's lengths.

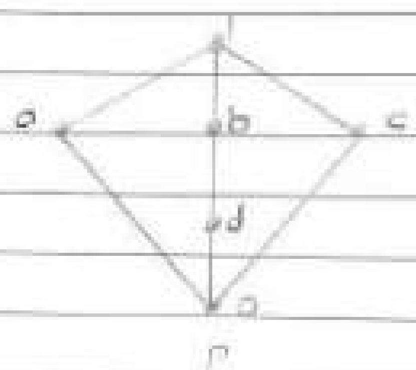
ii] Give some chains and it's lengths.

iii] Find length and width of Poset.

→ Let,

Given that,

A poset P defined by.



Chains	Length	Anti-chains	Length
{0, a, 1}	2	{a, b, c}	3
{0, d, b, 1}	3	{a, d, c}	3
{0, c, 1}	2	{a, c}	2
{0, d}	1	{b, c}	2
{0, b, 1}	2	{a, b}	2
{a, 1}	1	{a, d}	2
{d, b}	1	{d, c}	2

3] By def^n,

Length of Poset = Length of longest chain

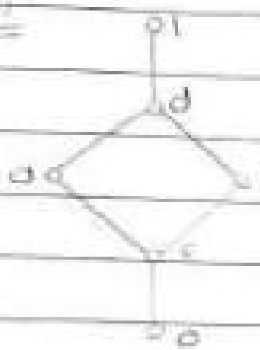
$\therefore$ Length of Poset = 3

Width of Poset = Length of longest antichain
= 3

Upper bound :-

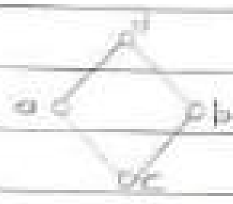
Let $\langle P \ \leq \rangle$ be a poset a non-empty subset H of P has an upper bound $a \in P$ if $h \leq a \quad \forall \, h \in H$

e.g.



P

Let,



H

Clearly, 1 and d are upper bounds of H.

Least Upper bound (lub / supremum) :-

Let $\langle P \ \leq \rangle$ be a poset a non-empty subset H of P has least upper bound $a \in P$ iff for every upper bound b of H we have $a \leq b$.

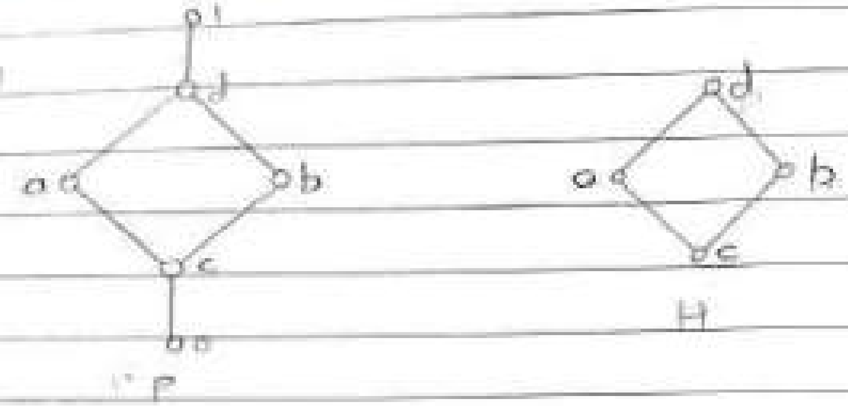
We denote $a = \sup H$

OR $a = \vee H$

Lower bound :-

Let $\langle P \ \leq \rangle$ be a poset a non-empty subset H of P has a lower bound $a \in P$ if $a \leq h \quad \forall \, h \in H$

e.g.



clearly, c and 0 are lower bounds of H.

Greatest lower bound (glb/ infimum):-

Let $\langle P \leq \rangle$ be a poset a non-empty subset H of P has greatest lower bound $a \in P$ iff for every lower bound b of H we have $b \leq a$.

We denote, $a = \text{Inf } H$

OR $a = \wedge H$

$\wedge$ meet

Duality Principle :-

If ϕ is a statement which is true in all posets then it's dual is also true in all posets.

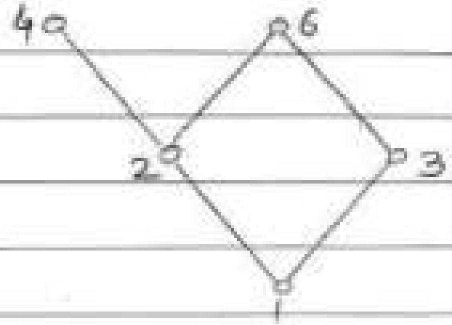
Semi-Lattice:-

A poset is called semi-lattice if infimum of any two elements or supremum of any two elements are exist.

Lattice:-

Let, a poset P is called lattice if supremum and infimum of any two elements ~~exist~~ of P exist.

1] Let. $P = \{1, 2, 3, 4, 6\}$ be a poset i.e. $(P, \le)$ is poset if $a \le b$ iff $a | b$.

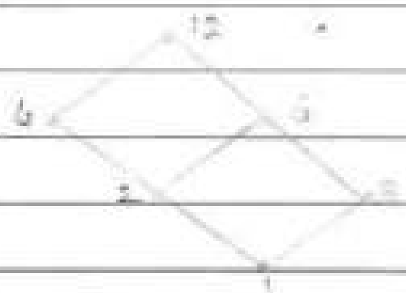


Clearly, P is semi-lattice because any two elements have infimum.

i.e. $\forall\ a, b \in P \Rightarrow \inf\{a, b\}$ exists.

But P is not lattice because supremum of 4 and 6 does not exists i.e. $\sup\{4, 6\}$ does not exists.

2] Let $P = \{1, 2, 3, 4, 6, 12\}$ be a poset i.e. $(P, \le)$ is poset if $a \le b$ iff $a | b$.



Clearly, P is lattice because $\forall\ a, b \in P \Rightarrow \sup\{a, b\}$ and $\inf\{a, b\}$ exists.

3) Every chain is Lattice.

→ **Proof:-**

Let, c be any chain.
Let, $a, b \in c$ be any elements.
By def^n of chain any two elements are comparable.
a, b must be comparable.
$\Rightarrow$ Either $a \le b$ or $b \le a$.

Case i) If $a \le b$
In this case,
$\sup\{a, b\} = b$ and $\inf\{a, b\} = a$
i.e. $\forall a, b \in c$ $\sup\{a, b\}$ and $\inf\{a, b\}$ exists.

Case ii) If $b \le a$
In this case,
$\sup\{a, b\} = a$ and $\inf\{a, b\} = b$
i.e. $\forall a, b \in c$ $\sup\{a, b\}$ and $\inf\{a, b\}$ exists.

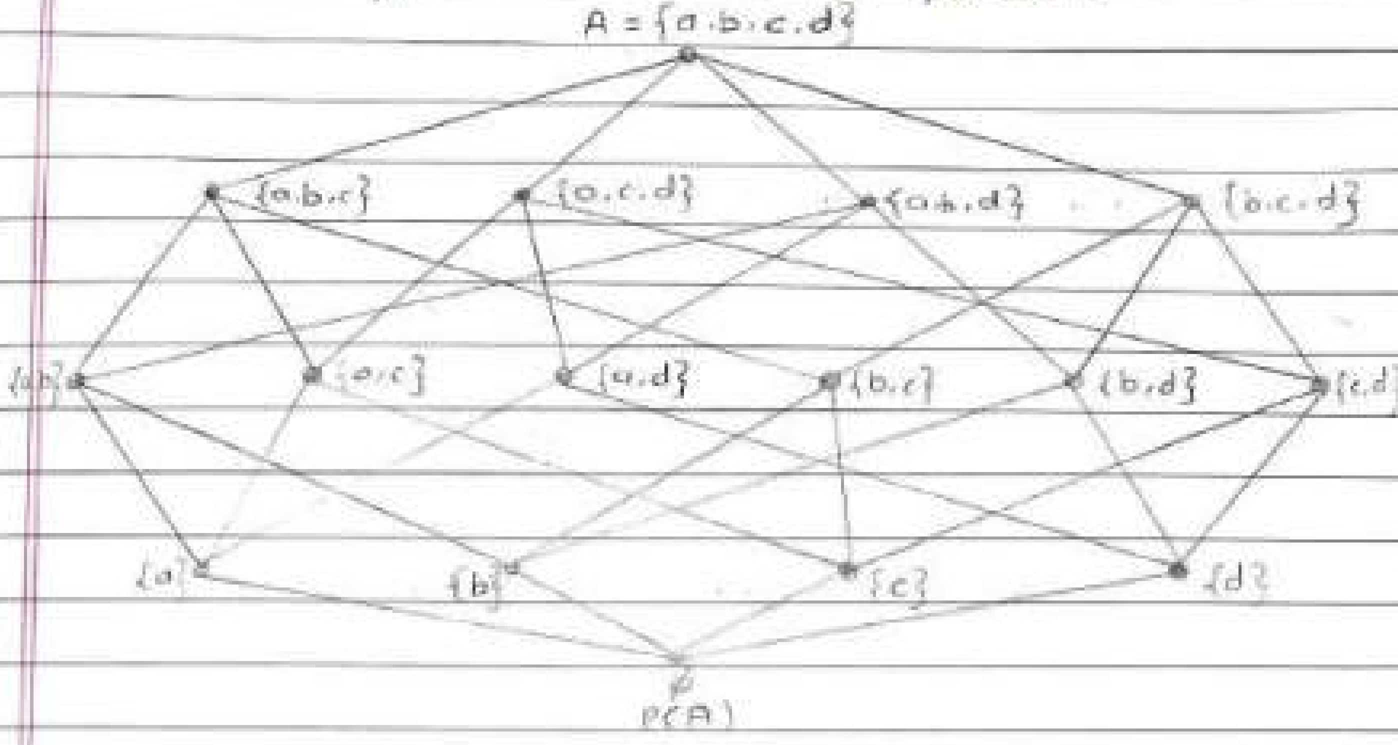
From (i) & (ii)
Every chain must be Lattice.

4) Power set of any set is Lattice under set inclusion.

→ Let,
$A = \{a, b, c, d\}$ be any set.

$\therefore P(A) = \{ \phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}, A \}$

Clearly, $\langle P(A) \ \subseteq \rangle$ is poset.

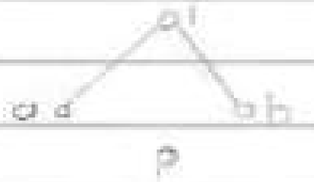


Clearly, it is lattice.

Note:-

1) Every lattice is poset, but converse may or may not be true.

e.g.



Clearly P is Poset but not lattice.

Th^m 1:-

A poset $\langle L \ \leq \rangle$ is a lattice iff sup H and inf H exists for any non-empty subset H of L i.e. $\Phi \neq H \subseteq L$ and H is finite.

→ Proof:-

Let, $\langle L \ \leq \rangle$ is Lattice.

⇒ By def^n sup{a, b} and inf {a, b} exists

for any two elements $a, b \in L$. ($\forall\ a, b \in L$)
We use induction on number of elements $\phi \neq H \subseteq L$.

If $H = \{a\}$
$\Rightarrow$ $\sup H = \{a\} = \inf H$

If $H = \{a, b\}$
$\Rightarrow$ $\sup H$ and $\inf H$ exists, by def^n of L

If $H = \{a, b, c\}$
$\Rightarrow$ $\sup \{a, b\}$ exists say 'k'.
Also $\sup\{k, c\}$ exists. say 't'.

<u>claim</u>:- $\sup H = t$
We have,
$a \leq t$, $b \leq t$, $c \leq t$
$\Rightarrow$ 't' is an upper bound of $H = \{a, b, c\}$

Let, t' is any upper bound of $H = \{a, b, c\}$
$\Rightarrow$ $a \leq t'$, $b \leq t'$, $c \leq t'$
We have,
t' is an upper bound of a and b.
But, k is $\sup\{a, b\}$ i.e. supremum of a and b. (least upper bound).
$\Rightarrow$ $\underline{k \leq t'}$.

$\Rightarrow$ t' is upper bound of c and k.
and we have, t is supremum of c and k (least upper bound).
$\Rightarrow$ $\underline{t \leq t'}$
$\Rightarrow$ t is supremum of a, b and c.

$\Rightarrow \sup H = t$

Now, assume sup H exists where $H = \{a_1, a_2, \ldots a_k\}$

Let,

$H = \{a_1, a_2, \ldots a_k, a_{k+1}\}$. to prove sup H exists.

By hypothesis, $\sup\{a_k, a_{k+1}\}$ exists.

$\Rightarrow \sup\{a_1, a_2, a_3 \ldots a_{k-1}\}, \sup\{a_k, a_{k+1}\}$ exists.

(By induction hypothesis).

$\Rightarrow \sup\{a_1, a_2, a_3, \ldots a_{k-1}, a_k, a_{k+1}\}$ exists.

Thus, by induction sup H exists for any finite subset H of L.

By duality principle, infimum of H exists. i.e. inf H exists.

Conversely,

Suppose, sup H and inf H exists for any finite subset H of L.

Inparticular. $H = \{a, b\}$.

$\Rightarrow \sup H = \sup\{a, b\}$ and
$\inf H = \inf\{a, b\}$ exists.

As a, b are arbitrary elements of L.

$\Rightarrow \sup\{a, b\}$ and $\inf\{a, b\}$ exists $\forall a, b \in L$

$\Rightarrow$ L is Lattice.

Hence the proof.

Def^n:- $\wedge \xrightarrow{\text{meet}} : L \times L \to L$ and $\vee \xrightarrow{\text{join}} : L \times L \to L$ be two binary operations on L. Then L is called

i) $\wedge$ semi-lattice if $a \wedge b$ exists $\forall\, a, b \in L$.

ii) $\vee$ semi-lattice if $a \vee b$ exists $\forall\, a, b \in L$.

Def^n:- (Lattice)

1) If 'L' is said to be lattice iff $a \vee b$ and $a \wedge b$ exists $\forall\, a, b \in L$.

2) Let L be a non-empty set with two binary operations join ($\vee$) and meet ($\wedge$) satisfying following properties:

i) Idempotent property :-

$a \vee a = a \qquad \forall\, a \in L$

$a \wedge a = a \qquad \forall\, a \in L$

ii) Commutative property :-

$a \vee b = b \vee a \qquad \forall\, a, b \in L$

$a \wedge b = b \wedge a \qquad \forall\, a, b \in L$

iii) Associative property :-

$a \vee (b \vee c) = (a \vee b) \vee c \qquad \forall\, a, b, c \in L$

$a \wedge (b \wedge c) = (a \wedge b) \wedge c \qquad \forall\, a, b, c \in L$

iv) Absorption property :-

$a \vee (a \wedge b) = a \qquad \forall\, a, b \in L$

$a \wedge (a \vee b) = a \qquad \forall\, a, b \in L$

Th^m 2:-

Let $\langle L, \leq \rangle$ be a lattice with $a \wedge b = \inf\{a, b\}$ and $a \vee b = \sup\{a, b\}$ then,

i) $a \wedge a = a$, $a \vee a = a$
ii) $a \wedge b = b \wedge a$, $a \vee b = b \vee a$
iii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, $a \vee (b \vee c) = (a \vee b) \vee c$
iv) $a \vee (a \wedge b) = a$, $a \wedge (a \vee b) = a$.

Proof:-

Let, $\langle L \leq \rangle$ be a lattice.

i) Let $a \in L$ be any element.
Thus, $a \wedge a = \inf\{a, a\} = a$
$a \vee a = \sup\{a, a\} = a$

ii) Let $a, b \in L$ be any elements.
Thus, $a \wedge b = \inf\{a, b\} = \inf\{b, a\} = b \wedge a$
lly, $a \vee b = \sup\{a, b\} = \sup\{b, a\} = b \vee a$

iii) Let $a, b, c \in L$ be any elements.
consider,
$$a \wedge (b \wedge c) = \inf\{a, b \wedge c\}$$
$$= \inf\{a, \inf\{b, c\}\}$$
$$= \inf\{a, b, c\}$$
$$= \inf\{\inf\{a, b\}, c\}$$
$$= \inf\{a \wedge b, c\}$$
$$a \wedge (b \wedge c) = (a \wedge b) \wedge c$$

Similarly,
consider,
$$a \vee (b \vee c) = \sup\{a, b \vee c\}$$
$$= \sup\{a, \sup\{b, c\}\}$$
$$= \sup\{a, b, c\}$$
$$= \sup\{\sup\{a, b\}, c\}$$
$$= \sup\{a \vee b, c\}$$
$$a \vee (b \vee c) = (a \vee b) \vee c$$

iv] Let $a, b \in L$ be any elements.

case 1] $a \leq b$

Thus, $\inf\{a, b\} = a$

$\Rightarrow a \wedge b = a$

and $\sup\{a, b\} = b$

$\Rightarrow a \vee b = b$

Now,

$$a \vee (a \wedge b) = \sup\{a, a \wedge b\}$$
$$= \sup\{a, a\}$$
$$= a \vee a$$
$$= a$$

Similarly,

$$a \wedge (a \vee b) = \inf\{a, a \vee b\}$$
$$= \inf\{a, b\}$$
$$= a \wedge b$$
$$= a$$

case 2] $b \leq a$

Thus, $\inf\{a, b\} = b$

$\Rightarrow a \wedge b = b$

and $\sup\{a, b\} = a$

$\Rightarrow a \vee b = a$

Now,

$$a \vee (a \wedge b) = \sup\{a, a \wedge b\}$$
$$= \sup\{a, b\}$$
$$= a \vee b$$
$$= a$$

Similarly,

$$a \wedge (a \vee b) = \inf\{a, a \vee b\}$$
$$= \inf\{a, a\}$$
$$= a \wedge a$$
$$= a$$

Thus, by case ① + ②,
$a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$.

2 Prove the following.
If $x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq x_0$ then
$x_0 = x_1 = x_2 = x_3 = \cdots = x_n \in P$ where
P is poset.

→ Let,
If $x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq x_0$
consider, 1st three terms.
$x_0 \leq x_1 \leq x_2$
$\Rightarrow x_0 \leq x_1$ and $x_1 \leq x_2$
$\Rightarrow x_0 \leq x_2$ ∵ (Transitivity)

consider,
$x_0 \leq x_2 \leq x_3$
$\Rightarrow x_0 \leq x_2$ and $x_2 \leq x_3$
$\Rightarrow x_0 \leq x_3$ ∵ (Transitivity)

continuing in this way, we get,
$\Rightarrow x_0 \leq x_n$
consider,
$x_0 \leq x_n$ and $x_n \leq x_0$
$\Rightarrow x_0 \leq x_n$ ∵ (Anti-symmetric)

Thus we get, $\because x_0 = x_n$
$x_0 \leq x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_{n-1} \leq x_0$
By using simillar argument we can prove
$x_{n-1} = x_0$
continuing in this way, we get.
$x_1 = x_2 = x_3 = \cdots = x_n = x_0$
Hence the result.

Th^m 3 :-

Let the algebra $\langle L, \wedge, \vee \rangle$ be a lattice and $a \le b$ iff $a = a \wedge b$ then $\langle L \le \rangle$ is poset and as a poset it is a ~~poset~~ lattice

$b = a \vee b$

→ **Proof** :-

Let, $\langle L \wedge \vee \rangle$ be a lattice.

Let, $a \le b$ iff $a = a \wedge b$.

To prove,

$\langle L \le \rangle$ is Poset.

1] Reflexivity :-

By Idempotent law we have,

$a \wedge a = a$

$\Leftrightarrow \quad a \le a \qquad \forall a \in L$

2] Antisymmetry :-

By commutative law we have,

$a \wedge b = b \wedge a$

Consider,

$a \le b$ and $b \le a$

$\Rightarrow a = a \wedge b$ and $b = b \wedge a$

$\Rightarrow \quad a = b$

(By commutative)

for some $a, b \in L$

3] Transitivity :-

Let $a \le b$ and $b \le c$

$\Leftrightarrow \quad a = a \wedge b$ and $b = b \wedge c \qquad$ — ①

By using associative property, we have,

$a \wedge (b \wedge c) = (a \wedge b) \wedge c$

$\Rightarrow a \wedge b = a \wedge c$

$\Rightarrow \quad a = a \wedge c \qquad$ —(By ①)

$\Rightarrow \quad a \le c \qquad$ —(∵ By ①)

Hence by ①, ② & ③,
$\langle L \leq \rangle$ is a Poset.

To prove $\langle L \leq \rangle$ is Lattice.
Let, $a, b \in L$ be any elements

Claim :- $\sup\{a,b\}$ and $\inf\{a,b\}$ exists.
Consider,

$(a \wedge b) \wedge b = a \wedge (b \wedge b)$ — By Associativity

$= a \wedge b$ — By Idempotent property

$\therefore (a \wedge b) \wedge b = a \wedge b$

$\therefore b \wedge (a \wedge b) = a \wedge b$ — By Commutativity

$\therefore a \wedge b \leq b$ —②

Similarly, consider

$(a \wedge b) \wedge a = a \wedge (a \wedge b)$ - commutativity

$= (a \wedge a) \wedge b$ - Associativity

$= a \wedge b$ - Idempotent

$\therefore (a \wedge b) \wedge a = a \wedge b$

$\therefore a \wedge b \leq a$ —③

By eq^n ② & ③,
$a \wedge b$ is lower bound of $\{a,b\}$.

Let, c be any other lower bound of $\{a,b\}$

$\Rightarrow c \leq a$ and $c \leq b$

$\Rightarrow c = c \wedge a$ and $c = c \wedge b$

Claim :- $c \leq a \wedge b$
Consider,

$$c \wedge (a \wedge b) = (c \wedge a) \wedge b$$
$$= c \wedge b$$
$$= c$$

i.e. $c = c \wedge (a \wedge b)$

$\Rightarrow \quad c \leq a \wedge b$

This shows that, $a \wedge b$ is greatest lower bound of $\{a, b\}$.

Hence, $\inf\{a, b\} = a \wedge b$ exists. $\forall a, b \in L$

By Duality principle,

$\sup\{a, b\} = a \vee b$ exists $\forall a, b \in L$

$\therefore \langle L \ \leq \rangle$ is Lattice.

Covering :- ($-\!\!<$)

If $a \leq b$ and there does not exist 't' such that $a \leq t \leq b$, then we called 'a' is covered by 'b' or 'b' covers 'a'.

Symbolically,

It is denoted by $a -\!\!< b$ or $a \prec b$.

Hasse Diagram :-

It is graphical representation of lattice poset, in which elements are denoted by small o and covering is denoted by '—'.

Note :-

1) Intersection of two lines gives a point in the plane but that does not always

mean a point of poset or lattice.

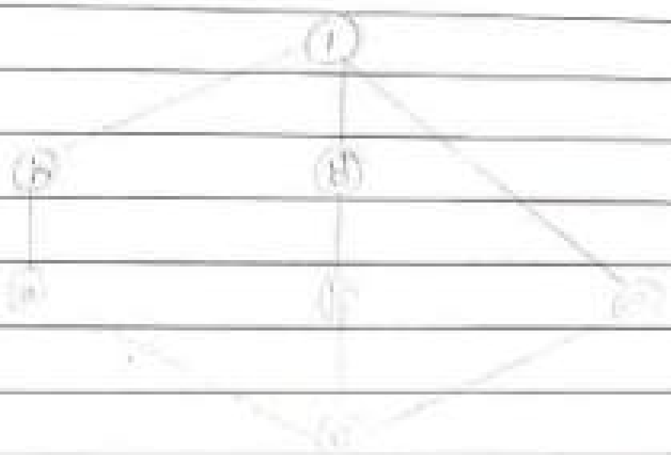
1) Using ~~dr~~ covering relations, draw Hasse diagrams

$0 \prec a \prec b \prec 1$

$0 \prec c \prec d \prec 1$

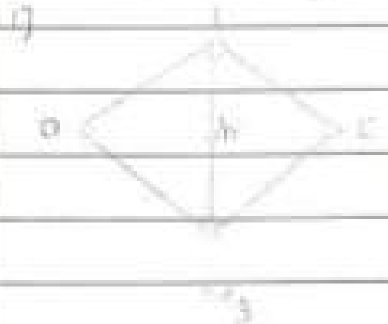
$0 \prec e \prec 1$

→ Let,



Hasse diagram

2) From the following Hasse diagrams, find the corresponding covering.

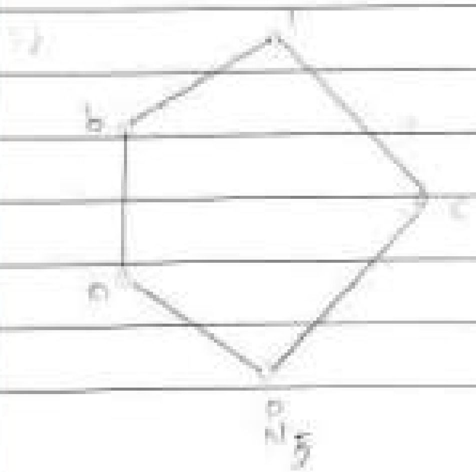


$0 \prec a \prec 1$

$0 \prec b \prec 1$

$0 \prec c \prec 1$

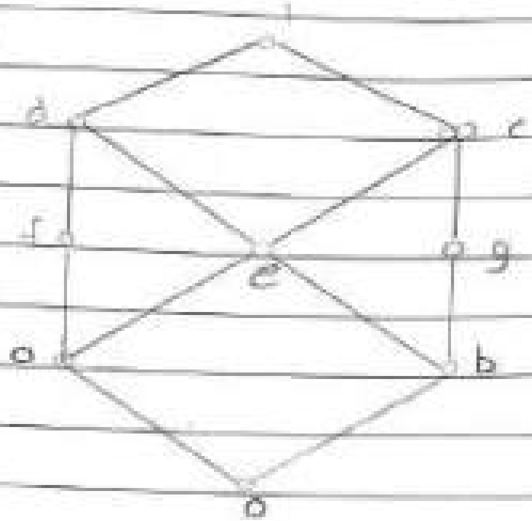
It is lattice.



$0 \prec a \prec b \prec 1$

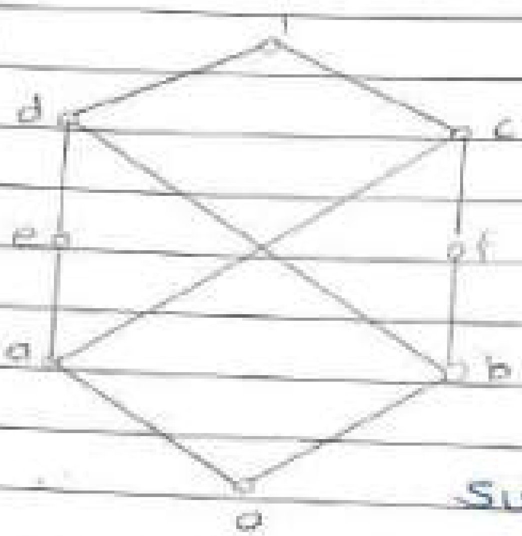
$0 \prec c \prec 1$

It is lattice.



0 ⋖ a ⋖ f ⋖ d ⋖ 1
0 ⋖ b ⋖ e ⋖ c ⋖ 1
0 ⋖ b ⋖ g ⋖ c ⋖ 1
0 ⋖ a ⋖ e ⋖ d ⋖ 1
0 ⋖ b ⋖ e ⋖ d ⋖ 1
0 ⋖ a ⋖ e ⋖ c ⋖ 1

It is Lattice.



0 ⋖ a ⋖ c ⋖ 1
0 ⋖ b ⋖ f ⋖ c ⋖ 1
0 ⋖ a ⋖ e ⋖ d ⋖ 1
0 ⋖ b ⋖ d ⋖ 1

It is not lattice bcoz Sup. and inf doesn't exist.

Lemma:-

Let, ⟨P ≤⟩ be a finite poset then $a \le b$ iff $a = b$ or $\exists$ a finite sequence of elements $x_1, x_2 \ldots x_n$ such that $a = x_1$, $b = x_n$ and each x_i is covered by x_{i+1}.

→ Proof:-

Let, ⟨P ≤⟩ be a finite poset.
~~& Let,~~ $\cancel{a} \neq b$

If $a = b$ then there is nothing to prove.

Suppose, $a \neq b$ then

As $a = b \Leftrightarrow a \leq b$

Suppose $a \neq b$.

Then $\exists$ a finite sequence of elements $x_1, x_2, \dots x_n$ such that,

$a = x_1 \prec x_2 \prec x_3 \prec x_4 \dots \prec x_n = b$

$\Rightarrow a = x_1 \leq x_2 \leq x_3 \leq x_4 \dots \leq x_n = b$

By applying transivity, we get,

$x_1 \leq x_2$ and $x_2 \leq x_3$

$\Rightarrow x_1 \leq x_3$ and $x_3 \leq x_4$

$\Rightarrow x_1 \leq x_4$ and $x_4 \leq x_5$

$\vdots$

$\Rightarrow x_1 \leq x_n$

$\Rightarrow a \leq b$

Conversely,

Let, $a \leq b$

$\Rightarrow$ Either $a = b$ or $a \neq b$

$\Rightarrow$ Either $a = b$ or $a < b$

If $a = b$ then we are done

If $a < b$

If a is covered by 'b' i.e. $a \prec b$. Then we have two elements in chain, and the result is true.

If a is not covered by 'b' then there exist $x_2 \in P$ such that,

$a \leq x_2 \leq b$

Assume, $a \prec x_2$ and $x_2 \leq b$.

If ~~x_2 &~~ $x_2 \prec b$ then we have a chain of three elements

i.e. $a \prec x_2 \prec b$ and the result is true.
Otherwise, $\exists\, x_3 \in P$ such that,

$$a \nleq a \prec x_2 \prec x_3 \leq b .$$

If $x_3 \prec b$ then we are done.
If not then continuing this process untill we reach at b which is covering atleast b and poset is finite. This process will stop after finite steps.

Thus, $\exists\, a = x_1, x_2, \ldots\ x_n = b \in P$ such that,

$$a = x_1 \prec x_2 \prec x_3 \ldots \prec x_n = b .$$

Hence the result.

Maximal Element:-

Let, $\langle P\ \leq \rangle$ be a poset, an element $x \in P$ is called maximal element of P if $x \leq y$ $(y \in P)$ then $x = y$.

Finite lattice ∃ unique maximal element

Minimal Element:-

poset ∃ minimal then ?

Let, $\langle P\ \geq \rangle$ be a poset, an element $x \in P$ is called minimal element of P if $x \geq y$ $(y \in P)$ then $x = y$.

Maximum Element:-

Let, $\langle P\ \leq \rangle$ be a poset, an element $x \in P$ is called maximum element of P if $a \leq x\ \forall\ a \in P$

Minimum element:-

Let, $\langle P, \leq \rangle$ be a poset, an element $x \in P$ is called minimum element of P if $x \leq a \quad \forall a \in P$.

Note:-

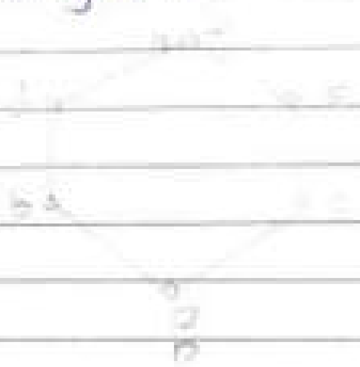
1) Every maximum element is maximal element, but converse may or may not be true.

e.g.

1) Consider the poset defined by following Hasse Diagram.

Clearly, d and e are maximal elements but maximum element does not exists.

2) Consider the poset defined by following Hasse Diagram.



Clearly, f is maximum and maximal element of P.

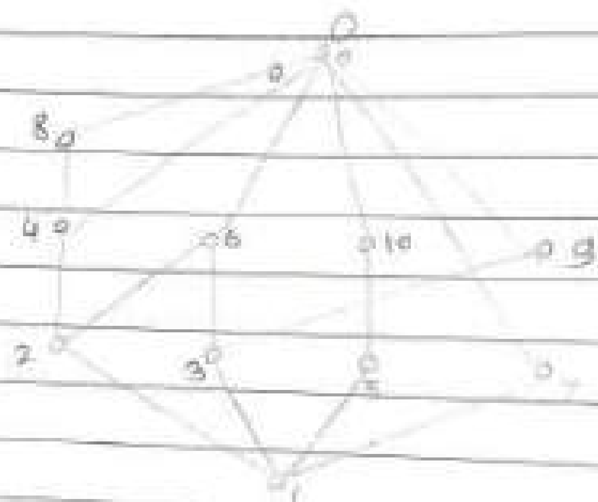
2) Every minimum element is minimal but converse may or may not be true.

Q) Find the maximal, minimal, maximum, minimum elements of the poset P defined by $P = \{0, 1, 2, \dots 10\}$ with $a \le b$ iff $a | b$.

→ Let,

Given that,

$P = \{0, 1, 2, \dots 10\}$ with $a \le b$ iff $a | b$.



Here,

0 is maximum element in P hence it is maximal also.

1 is minimum element in P hence it is minimal also.

Note:-

1) Every poset has unique maximum and minimum element if exists.

Ascending Chain Condition (ACC):-

A poset $\langle P \leq \rangle$ is said to be satisfy ascending chain condition (ACC) if any increasing chain terminates.

i.e. $x_0 \leq x_1 \leq x_2 \leq x_3 \leq \dots \leq x_n \leq x_{n+1} \leq \dots$

then $x_n = x_{n+1} = x_{n+2} = \dots$ for some $n \in \mathbb{W}$.

Descending Chain Condition (DCC):-

A poset $\langle P \leq \rangle$ is said to be satisfy descending chain condition (DCC) if any decreasing chain terminates.

i.e. $x_0 \geq x_1 \geq x_2 \geq x_3 \geq \dots \geq x_k \geq x_{k+1} \geq \dots$

then $x_k = x_{k+1} = x_{k+2} = \dots$ for some $k \in \mathbb{W}$.

Zorn's Lemma?:-

Statement:-

Let, $\langle P \leq \rangle$ be a poset such that every chain in P has an upper bound then P has maximal element.

Th^m 4:-

If a poset satisfies ACC then it has a maximal element.

→ Proof:-

Let $\langle P \leq \rangle$ be a poset which satisfies ascending chain condition (ACC).

Let, $x_0 \in P$ be any element.

If x_0 is maximal element then we are through.

If x_0 is not maximal element then there exist $x_1 \in P$ such that.

$$x_0 \leq x_1$$

If x_1 is maximal element then we are done.

If x_1 is not maximal element then there exist $x_2 \in P$ such that,

$$x_0 \leq x_1 \leq x_2$$

Continuing this process we get an increasing chain of elements in P say

$$x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots$$

Since, P satisfies ACC this chain must be terminate.

Therefore,

$\exists\ i \in W$ such that

$$x_i = x_{i+1} = x_{i+2} = \cdots$$

Thus, there will be no element in P this chain greater than x_i.

If this x_i covers all the elements of P then x_i is maximal element of P.

If not then there exist $y_0 \in P$ and $y_0 \neq x_i$ such that, $x_i \leq y_0$

If y_0 is maximal then we are done.

If y_0 is not maximal element then there exist $y_1 \in P$ such that,

$$x_i \leq y_0 \leq y_1$$

If y_1 is maximal element then we stop.

If y_1 is not maximal then continuing this process we have an increasing chain,

$$x_i \leq y_0 \leq y_1 \leq y_2 \leq \cdots \leq y_k \leq y_{k+1} \leq \cdots$$

Since P satisfies ACC this chain must be terminate i.e.

$\exists\ j \in N$ such that

$y_j = y_{j+1} = y_{j+2} = \cdots$

If y_j is all elements of P covered by y_j then y_j is maximal element.

otherwise y_j is upper bound of this chain.

Do the same argument for all possible chains.

Thus, every chain of poset P has an upper bound. Therefore by Zorn's Lemma P has maximal element.

Th^m 5:-

If a poset satisfies DCC then it has a minimal element.

→ Proof:-

We know that if poset satisfies ACC then it has maximal element.

By Duality principle we get required result.

i.e. If a poset satisfies DCC then it has minimal element.

Homomorphism:-

Let $\langle L_1, \wedge_1, \vee_1 \rangle$ and $\langle L_2, \wedge_2, \vee_2 \rangle$ be two Lattices and $\phi : L_1 \longrightarrow L_2$ is called homomorphism if,

1] $\phi(a \vee_1 b) = \phi(a) \vee_2 \phi(b) \quad \forall\ a, b \in L_1$

2] $\phi(a \wedge_1 b) = \phi(a) \wedge_2 \phi(b) \quad \forall\ a, b \in L_1$.

Note:-

1] Let $\langle L_1 \wedge_1 \vee_1 \rangle$ and $\langle L_2 \wedge_2 \vee_2 \rangle$ be two Lattices and $\phi : L_1 \rightarrow L_2$ is called join homomorphism if,

$$\phi(a \vee_1 b) = \phi(a) \vee_2 \phi(b) \qquad \forall a, b \in L_1$$

2] Let $\langle L_1 \wedge_1 \vee_1 \rangle$ and $\langle L_2 \wedge_2 \vee_2 \rangle$ be two Lattices and $\phi : L_1 \rightarrow L_2$ is called meet homomorphisn if,

$$\phi(a \wedge_1 b) = \phi(a) \wedge_2 \phi(b) \qquad \forall a, b \in L_1$$

3] ϕ is homomorphism then it is both meet and join homomorphism.

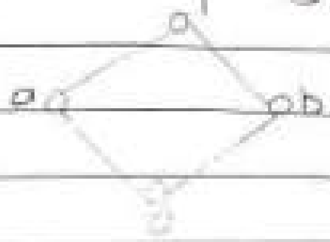
Isotone Map:- (order Preserving Map)

Let, $\langle P_1 \leq_1 \rangle$ and $\langle P_2 \leq_2 \rangle$ be two posets a map $\phi : P_1 \longrightarrow P_2$ such that $a \leq_1 b$ in P_1 iff $\phi(a_1) \leq_2 \phi(b)$ in P_2, is called isotone map.

Note:-

1] Homomorphism in Lattices is isotone map but converse may or may not be true.

1] Consider the Lattices defined by following Hasse Diagram.



P_1

P_2

We define $\phi : P_1 \to P_2$ by, $\phi(0) = 0$

$\phi(0) = 0$

$\phi(a) = c$

$\phi(b) = c$

$\phi(1) = 1$

observe that,

In P_1	In P_2
$0 \le a$	$\phi(0) \le \phi(a) \Leftrightarrow 0 \le c$
$0 \le b$	$\phi(0) \le \phi(b) \Leftrightarrow 0 \le c$
$0 \le 1$	$\phi(0) \le \phi(1) \Leftrightarrow 0 \le 1$
$a \le 1$	$\phi(a) \le \phi(1) \Leftrightarrow c \le 1$
$b \le 1$	$\phi(b) \le \phi(1) \Leftrightarrow c \le 1$

$\therefore$ $\phi : P_1 \to P_2$ is order preserving or isotone map.

Now consider,

$\phi(a \vee b) = \phi(1) = 1$

and

$\phi(a) \vee \phi(b) = c \vee c = c$

$\therefore$ $\phi(a \vee b) \neq \phi(a) \vee \phi(b)$ $\quad a, b \in P_1$

Similarly,

$\phi(a \wedge b) = \phi(0) = 0$

and

$\phi(a) \wedge \phi(b) = c \wedge c = c$

$\therefore$ $\phi(a \wedge b) \neq \phi(a) \wedge \phi(b)$ $\quad a, b \in P_1$

$\therefore$ ϕ is neither join nor meet homomorphic

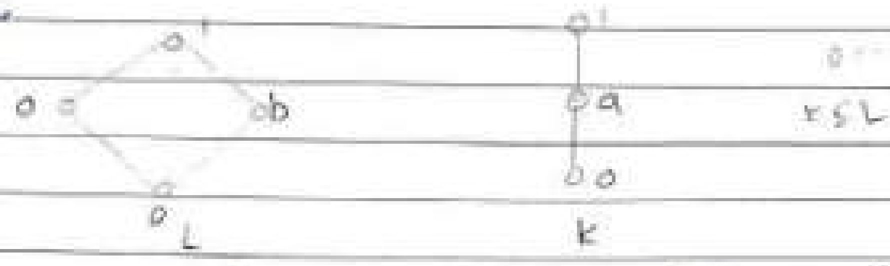
Sublattice :-

Def^n 1 :-

Let, K be a non-empty subset of lattice 'L' (lattice as a poset). If K is subposet K is a lattice then K is called sublattice of L.

Eg.

Consider the lattices L and K defined by.



Clearly, K is sub-lattice of L.

Def^n 2 :-

A non-empty subset K of L i.e. $\phi \neq K \subseteq L$ and $\langle L, \wedge, \vee \rangle$ is lattice then K is called sublattice of L if $\langle K \wedge \vee \rangle$ is lattice.

Ideal :-

Let $\langle L \wedge \vee \rangle$ be a lattice a non-empty subset I of L ($\phi \neq I \subseteq L$) is called ideal of L if,

i) I is sublattice of L

ii) For $a \in I$ and $l \in L$ then $a \wedge l \in I$

Proper ideal :-

If L be a lattice and I is ideal of L then I is called proper ideal if $I \neq$

If $I = \{0\}$ then I is called trivial ideal.

Prime ideal :-

A proper ideal I is called prime ideal iff for $a, b \in L$ and $a \wedge b \in I$,
$\Rightarrow$ either $a \in I$ or $b \in I$.

Principle Ideal:-

Ideal generated by single element is called principle ideal.

Note:-

1] Ideal generated by 'a' is denoted by $(a]$.

2] Consider the lattice defined by.

Now we define subsets of L.

Subsets of L	Ideal	Proper Ideal	Prime Ideal	Principle Ideal
1] $I_1 = \{0\}$	Yes	Yes	No Bcoz, Let, $a, b \in L$ $\Rightarrow a \wedge b = 0 \in I_1$ but neither $a \in I_1$ or $b \in I_1$	Yes $\because (0] = I_1$
2] $I_2 = \{0, a\}$	Yes	Yes	Yes	Yes $\because (a] = I_2$

Subsets of L	Ideal	Proper Ideal	Prime Ideal	Principle Ideal
3] $I_3 = \{0, b\}$	Yes	Yes	Yes	Yes $\because (b] = I_3$
4] $I_4 = L$	Yes	No $\because$ It is improper	No $\because$ It is improper	Yes $(1] = I_4$
5] $I_5 = \{0, a, b\}$	No $\because I_5$ is not sublattice.	No	No	No
6] $I_6 = \{0, b, 1\}$	No $\because a \in L, 1 \in I_6$ $a \wedge 1 = a \notin I_6$ but it is a sublattice.	No	No	No
7] $I_7 = \{a\}$	No $\because 0 \in L, a \in I_7$ $0 \wedge a = 0 \notin I_7$ but it is a sublattice	No	No	No

Remark:-

Prime ideal:-

A proper ideal I is called prime ideal iff for $a, b \in L$, $a, b \notin I \Rightarrow a \wedge b \notin I$.

e.g.

Lemma 3 :-

Let 'L' be a lattice and let I be a non-empty subset of 'L' then I is called ideal iff,

1] $a, b \in I \Rightarrow a \vee b \in I$

2] $a \in I$, $x \in L$, $x \leq a \Rightarrow x \in I$.

→ Proof :-

Let, 'L' be a lattice.

Let, $\phi \neq I \subseteq L$

Suppose I is ideal.

i] As I is ideal of L.

$\Rightarrow$ I is sublattice of L.

$\Rightarrow$ For any $a, b \in I \Rightarrow a \vee b \in I$

ii] Let $x \in L$, $a \in I$ be any elements such that $x \leq a$,

Claim :- $x \in I$

As $x \in L$, $a \in I$

$\Rightarrow x \wedge a \in I$ (∵ I is ideal)

$\Rightarrow x \in I$ (∵ $x \leq a$).

Conversely,

Let, 1] $a, b \in I \Rightarrow a \vee b \in I$

2] $a \in I$, $x \in L$. $x \leq a \Rightarrow x \in I$

are satisfied.

Claim :- I is ideal.

i] Let $a, b \in I$ be any elements.

First we prove. I is sublattice of L.

As, $a, b \in I \Rightarrow a \vee b \in I$ (∵ By assumption ①)

Also, we know that,

$a \wedge b \le a$

Let,

$a \in I$, $a \wedge b \in L$, $a \wedge b \le a$

$\Rightarrow a \wedge b \in I$ (∵ By assumption ②)

∴ I is sublattice of L. —— Ⓘ

ii) Let, $a \in I$, $l \in L$ be any elements.

$\Rightarrow a \wedge l \le a$

As, $a \in I$, $l \in L$, $a \wedge l \le a$

$\Rightarrow a \wedge l \le a$ (∵ By assumption ②)

$\Rightarrow a \wedge l \in I$ —— Ⓘⓘ

∴ By Ⓘ & Ⓘⓘ

I is ideal of L.

1) Show that a non-empty subset I of a lattice L (i.e. $\phi \ne I \subseteq L$) is an ideal of L iff.

$a, b \in L$, $a \vee b \in I \Rightarrow a, b \in I$.

→ Proof :-

Let, $\phi \ne I \subseteq L$, where 'L' is lattice

Let, I is ideal of L.

Consider,

$a, b \in L$ such that $a \vee b \in I$

Claim :- $a, b \in I$

We know that,

$a \le a \vee b$

$\Rightarrow a = a \wedge (a \vee b) \in I$ (∵ by using absorption law of sublattice)

$\Rightarrow a \in I$

Similarly,

$b \le a \vee b$ $\quad b \wedge b = b \wedge (a \vee b)$, $b = b \wedge (a \vee b)$

$\Rightarrow \quad b = b \wedge (a \vee b) \in I$

$\Rightarrow \quad b \in I$

$\therefore \quad a, b \in I$

Conversely,

Let, $a, b \in I$ such that $a \vee b \in I$

$\Rightarrow \quad a, b \in I$

$a \le a \vee b$

$a \le b$

$a \wedge b$

$a \le a$

$b \le a \vee b$

claim :- I is ideal.

As,

$a, b \in I \quad , \quad a \vee b \in I$

As, $\quad a = a \vee (a \wedge b) \in I$

$\Longrightarrow \quad a \wedge b \in I$

$\therefore$ I is sublattice of L.

Let, $x \in L$, $a \in I$ be any elements.

$\Rightarrow a \le a \vee x$

$\Rightarrow \quad x \in I$

$\therefore$ I is ideal of L.

Lemma 4:-

Let L be a lattice and H, I be non-empty subsets of L then I is generated by H, iff the following three conditions holds. (i.e. $I = (H]$)

i) I is ideal

ii) $H \subseteq I$

iii) $\forall \; i \in I$ $\exists$ an integer $n \ge 1$ and there exist elements $h_0, h_1, h_2, \ldots h_{n-1} \in H$ such that $i \le h_0 \vee h_1 \vee h_2 \ldots \vee h_{n-1}$.

→ Proof:-

Let, L be a lattice.

Let, H and I be non-empty subsets of L.

Suppose the three conditions are holds.

Let,

$I_0 = \{ x \in L \mid x \le h_0 \vee h_1 \vee h_2 \dots \vee h_{n-1},\ h_i \in H,\ i = 1, 2, \dots n-1 \}$

Claim:- I_0 is smallest ideal containing H

By def^n of I_0, we get,

H is contained in I_0.

$\Rightarrow \quad H \subseteq I_0$

Let, $x_1, x_2 \in I_0$ be any elements.

$\Rightarrow x_1 \le h_0 \vee h_1 \vee h_2 \dots \vee h_{m-1} = t$

$x_2 \le h_0 \vee h_1 \vee h_2 \dots \vee h_{m-1} = t$

∴ 't' is an upper bound of x_1 and x_2.

but, we know that,

$x_1 \vee x_2 = \sup\{x_1, x_2\}$ = least upper bound of x_1 and x_2.

$\therefore \quad x_1 \vee x_2 \le t = h_0 \vee h_1 \vee h_2 \vee \dots \vee h_{n-1}.$

$\therefore \quad x_1 \vee x_2 \le h_0 \vee h_1 \vee h_2 \vee \dots \vee h_{n-1}.$

$\Rightarrow \quad x_1 \vee x_2 \in I_0 \qquad \forall x_1, x_2 \in I_0$

—— ①.

Also, we have,

$x_1 \wedge x_2 \le x_1 \le h_0 \vee h_1 \vee h_2 \vee \dots \vee h_{n-1} = t$

$\Rightarrow x_1 \wedge x_2 \le t$

$\Rightarrow x_1 \wedge x_2 \le h_0 \vee h_1 \vee h_2 \vee \dots \vee h_{n-1}$

$\Rightarrow \quad x_1 \wedge x_2 \in I_0 \qquad \forall x_1, x_2 \in I_0$

—— ②

By eq^n ① & ②,

I_0 is sublattice of L. —③

Let, $l \in L$, $x \in I_0$ be any elements.

Consider,

$$x \wedge l \leq x \leq t$$

$$\Rightarrow x \wedge l \leq t \quad \text{(By transitivity)}$$

$$\Rightarrow x \wedge l \leq h_0 \vee h_1 \vee h_2 \vee \ldots \vee h_{n-1}$$

$$\Rightarrow x \wedge l \in I_0 \quad \forall x \in I_0 \ \& \ \forall l \in L. \quad \text{—④}$$

By ③ & ④,

I_0 is an ideal of L.

Clearly, $H \subseteq I_0$.

Now, consider I_1 is an ideal containing H

<u>Subclaim</u>:- I_0 is contained in I_1. ($I_0 \subseteq I_1$).

Let,

$x \in I_0$ be any element.

$\Rightarrow x \leq h_0 \vee h_1 \vee h_2 \vee \ldots \vee h_{n-1}$

As,

$$H \subseteq I_1$$

$$\Rightarrow h_0 \vee h_1 \vee h_2 \vee \ldots \vee h_{n-1} = t \in I_1$$

$$\Rightarrow x \in I_0 \subseteq L \text{ and } t \in I_1$$

$$\Rightarrow x \wedge t \in I_1$$

$$\Rightarrow x \in I_1$$

$$\Rightarrow I_0 \subseteq I_1$$

$\therefore$ I_0 is smallest ideal containing H.

Clearly,

$$\underline{I_0 = (H]}.$$

Conversely,

Let $I = (H]$ then three conditions are trivially holds.

Note:-

1] $I(L)$ is a/the set of all ideals of lattice L.

2] $I(L)$ is lattice under the set inclusion. It is called ideal lattice.

3] $(a] = \{x \in L \mid x \leq a\}$

Q. Show that $I(L)$ is lattice under the set inclusion

→ Proof:-

$I(L)$ = set of all ideals of lattice L.

$\Rightarrow I(L) = \{ \phi \neq I \subseteq L \mid I \text{ is ideal} \}$

$\{0\}$ & L are always lattice

Clearly, $I(L)$ is non-empty (since $\{0\}$, L are always ideals of lattice).

Let, $I_1, I_2 \in I(L)$ be any elements.

case I] $I_1 \subseteq I_2$

$\therefore \sup\{I_1, I_2\} = I_1 \cup I_2 = I_2$

Similarly,

$\operatorname{Inf}\{I_1, I_2\} = I_1 \cap I_2 = I_1$

$\therefore I(L)$ is lattice in this case.

case II] $I_2 \subseteq I_1$

$\sup\{I_1, I_2\} = I_1 \cup I_2 = I_1$

Similarly,

$Inf\{I_1, I_2\} = I_1 \cap I_2 = I_2$

$\therefore$ $I(L)$ is lattice in this case.

Case III] Neither $I_1 \nsubseteq I_2$ nor $I_2 \nsubseteq I_1$

$Inf\{I_1, I_2\} = I_1 \cap I_2$

Clearly, $I_1 \cap I_2$ is ideal.

$\Rightarrow$ $Inf\{I_1, I_2\} \in I(L)$.

Now, here we define

$Sup\{I_1, I_2\}$

$I_1 \vee I_2 \nleq (a]$ for some $a \in L$

$\therefore$ $Sup\{I_1, I_2\} = (a] \in I(L)$ such that

$I_1 \cup I_2 \subseteq (a]$. for some $a \in L$.

$\therefore$ By case I, II & III,

$Sup\{I_1, I_2\}$, $Inf\{I_1, I_2\}$ exists for all $I_1, I_2 \in I(L)$.

This shows that,

$I(L)$ is lattice.

Q. Show that $(a] \wedge (b] = (a \wedge b]$.

$\rightarrow$ Proof:-

Let, $x \in (a] \wedge (b]$

$\Leftrightarrow$ $x \in (a]$ and $x \in (b]$

$\Leftrightarrow$ $x \leq a$ and $x \leq b$

$\Leftrightarrow$ $x \wedge x \leq a \wedge b$

$\Leftrightarrow$ $x \leq a \wedge b$

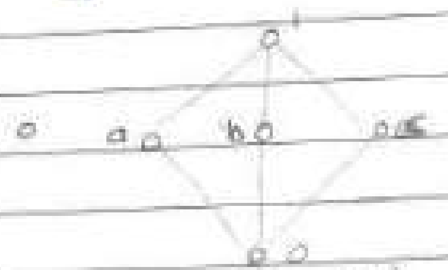
$\Leftrightarrow$ $x \in (a \wedge b]$

$\therefore$ $(a] \wedge (b] = (a \wedge b]$

Q. Show that $(a] \vee (b]$ may or may not be equal to $(a \vee b]$ with counter example.

→ Let.

consider the lattice defined by following Hasse diagram.



M_3 (L)

Here, $(a] = \{0, a\}$
$(b] = \{0, b\}$

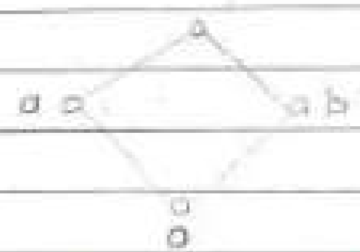
Here, $a \vee b = 1$

Thus, $(a \vee b] = L = M_3$

But, $(a] \vee (b] = \{0, a, b, 1\}$

$\therefore (a] \vee (b] \neq (a \vee b]$

Now consider the lattice defined by,



Here, $(a] = \{0, a\}$
$(b] = \{0, b\}$

Here, $a \vee b = 1$

Thus, $(a \vee b] = L$

and $(a] \vee (b] = L$

$\therefore (a] \vee (b] = (a \vee b]$

Hence the result.

Note :-

1] $I_0(L) = I(L) \cup \{\phi\}$

Q. A finite lattice L can be embedded in $I(L)$ and also in $I_0(L)$.

→ Let,

L be a finite lattice and $I(L)$ is the set of all ideals (ideal lattice).

Claim :- $\exists$ homomorphism from $L \to I(L)$ which is one-one.

Let, $\phi : L \to I(L)$ defined by,

$\phi(a) = (a]$.

Let, $a, b \in L$ be any elements.

consider,

$a = b$

$\Leftrightarrow (a] = (b]$

$\Leftrightarrow \phi(a) = \phi(b)$

$\therefore$ ϕ is well-defined and one-one.

Subclaim :- ϕ is homomorphism.

1] Let $\phi(a \wedge b) = (a \wedge b]$

$\Rightarrow \phi(a \wedge b) = (a] \wedge (b]$

$\Rightarrow \phi(a \wedge b) = \phi(a) \wedge \phi(b)$ $\quad \forall a, b \in L$

$\therefore$ ϕ is meet homomorphism.

2] Let $\phi(a \vee b) = (a \vee b]$

By absorption property in lattice L

$a = a \wedge (a \vee b)$

$\Rightarrow \phi(a) = \phi\,[a \wedge (a \vee b)]$

$\Rightarrow \phi(a) = \phi(a) \wedge \phi(a \vee b)]$ (∵ ϕ is meet homo)

$\Rightarrow \phi(a) = \phi(a) \wedge \phi(a \vee b)$ —— ①

Now again by absorption property in $I(L)$,

$$\phi(a) = \phi(a) \wedge [\phi(a) \vee \phi(b)] \quad \text{—— ②}$$

By ① + ②,

$$\phi(a) \wedge \phi(a \vee b) = \phi(a) \wedge [\phi(a) \vee \phi(b)]$$

$$\Rightarrow \quad \phi(a \vee b) = \phi(a) \vee \phi(b)$$

∴ ϕ is join-homomorphism.

∴ ϕ is homomorphism, and which is one-one i.e. ϕ is embedding mapping.

Similarly, a finite lattice L can be embedded in $I_0(L)$.

Hence the result.

Lemma 5:-

Let I is proper ideal of lattice 'L' iff there is a join homomorphism ϕ of L onto C_2 such that $I = \phi^{-1}(0)$ i.e. $I = \{x \in L \mid \phi(x) = 0\}$, where $C_2 = \{0, 1\}$ w.r.to $\oplus_2$, is cyclic group.

→ **Proof:-**

Suppose, I is proper ideal of lattice 'L'.

Define a map $\phi : L \to C_2$ by.

$$\phi(x) = \begin{cases} 0 & \text{if} \quad x \in I \\ 1 & \text{if} \quad x \notin I \end{cases}$$

Since I is proper ideal of L, and clearly ϕ is onto.

Let, $a, b \in L$ be any elements.

<u>case 1]</u> $a \vee b \in I$, $a, b \in I$

$\therefore \phi(a \vee b) = 0$ and $\phi(a) = 0$, $\phi(b) = 0$

But we have,

$0 = 0 \vee 0$

$\Rightarrow \phi(a \vee b) = \phi(a) \vee \phi(b)$

$\therefore$ ϕ is join homomorphism.

<u>case 2]</u> $a \vee b \notin I$, $a \in I$ & $b \notin I$ / $a \notin I$, $b \in I$

$\therefore \phi(a \vee b) = 1$

Also,

$\phi(a) = 0$, and $\phi(b) = 1$

But, $1 = 0 \vee 1$

$\Rightarrow \phi(a \vee b) = \phi(a) \vee \phi(b)$

$\therefore$ ϕ is join homomorphism.

<u>case 3]</u> $a \vee b \notin I$, $a, b \notin I$

$\therefore \phi(a \vee b) = 1$

Also,

$\phi(a) = 1$ and $\phi(b) = 1$

But, $1 = 1 \vee 1$

$\Rightarrow \phi(a \vee b) = \phi(a) \vee \phi(b)$

$\therefore$ ϕ is join homomorphism.

By case ①, ② & ③, ϕ is join homomorphism

Clearly, by def^n of ϕ,
$$I = \{x \in L \mid \phi(x) = 0\}$$
i.e.
$$I = \phi^{-1}(0)$$

Conversely,
Let ϕ is join homomorphism and $I = \{x \in L \mid \phi(x) = 0\}$ i.e. $I = \phi^{-1}(0)$.

Let $a, b \in I$ be any elements.
$\Rightarrow \quad \phi(a) = 0$ and $\phi(b) = 0$

Also,
$\phi(a \vee b) = \phi(a) \vee \phi(b)$ $\quad \because$ (ϕ is join homomorphism)
$\Rightarrow \quad \phi(a \vee b) = 0 \vee 0$
$\Rightarrow \quad \phi(a \vee b) = 0$
$\Rightarrow \quad a \vee b \in I \quad \forall a, b \in I$

Now, by using absorption property,
$a = a \vee (a \wedge b)$
$\Rightarrow \quad \phi(a) = \phi[(a) \vee (a \wedge b)]$ $\quad \because$ ϕ is well defined.
$\Rightarrow \quad 0 = \phi(a) \vee \phi(a \wedge b)$ $\quad \because a \in I$

$\Rightarrow \quad 0 = 0 \vee \phi(a \wedge b)$
$\Rightarrow \quad 0 = \phi(a \wedge b)$
$\Rightarrow \quad a \wedge b \in I \quad \forall a, b \in I$

$\therefore$ I is sublattice of lattice L.

Now consider $a \in I$ be any element & $x \in L$ be any element.
$\Rightarrow \quad \phi(a) = 0$

Now, by absorption property.

$$a = a \vee (a \wedge x)$$
$$a = a \vee (a \wedge x)$$
$$\Rightarrow \phi(a) = \phi[a \vee (a \wedge x)]$$

$$\Rightarrow 0 = \phi(a) \vee \phi(a \wedge x)$$
$$\Rightarrow 0 = 0 \vee \phi(a \wedge x)$$
$$\Rightarrow 0 = \phi(a \wedge x)$$
$$\Rightarrow a \wedge x \in I \qquad \forall a \in I, \forall x \in L$$

$\therefore$ I is an ideal of L.

As ϕ is onto join homomorphism.
$\Rightarrow$ I is proper ideal of L.

Lemma 6:-
I is prime ideal iff there is homomorphism of L onto C_2 with $\phi^{-1}(0) = I$ i.e. $I = \{x \in L / \phi(x) = 0\}$

→ Proof:-
Suppose I is prime ideal of L.

Define a map $\phi : L \to C_2$ by.
$$\phi(x) = \begin{cases} 0 & x \in I \\ 1 & x \notin I \end{cases}$$

Since I is prime ideal.
$\Rightarrow$ I is proper ideal.
$\therefore$ ϕ is onto map.

Claim:- ϕ is homomorphism.
Subclaim:- ϕ is join homomorphism.

Case 1) $a \vee b \in I$ with $a, b \in I$

$\Rightarrow \phi(a \vee b) = 0 = 0 \vee 0 = \phi(a) \vee \phi(b)$

$\therefore$ ϕ is join homomorphism.

Case 2) $a \vee b \notin I$, $a \in I$, $b \notin I$

$\Rightarrow \phi(a \vee b) = 1 = 0 \vee 1 = \phi(a) \vee \phi(b)$

$\therefore$ ϕ is join homomorphism.

Case 3) $a \vee b \notin I$, $a, b \notin I$

$\Rightarrow \phi(a \vee b) = 1 = 1 \vee 1 = \phi(a) \vee \phi(b)$

$\therefore$ ϕ is join homomorphism.

By case ①, ② & ③,

ϕ is join homomorphism.

Subclaim :- ϕ is meet homomorphism.

As I is prime ideal.

Let, $a, b \in L$ be any elements such that

$a \wedge b \in I$

$\Rightarrow$ Either $a \in I$ or $b \in I$

Now,

$\phi(a \wedge b) = 0$ with $a \in I \Rightarrow \phi(a) = 0$

$\Rightarrow \therefore \phi(a \wedge b) = 0 = 0 \wedge 1 = \phi(a) \wedge \phi(b)$.

OR

$\phi(a \wedge b) = 0$ with $b \in I \Rightarrow \phi(b) = 0$

$\therefore \phi(a \wedge b) = 0 = 1 \wedge 0 = \phi(a) \wedge \phi(b)$

$\therefore$ ϕ is meet homomorphism.

$\therefore$ ϕ is homomorphism.

Clearly, by def^n of ϕ,
$I = \{ x \in L \mid \phi(x) = 0 \} = \phi^{-1}(0)$.

Conversely,
Suppose that there is a homomorphism $\phi : L \xrightarrow{\text{onto}} C_2$ with $I = \phi^{-1}(0)$, $I = \{ x \in L \mid \phi(x) = 0 \}$

Claim:- I is prime ideal.
Subclaim:- I is proper ideal.

Let $a, b \in I$ be any elements.
$\Rightarrow \phi(a) = 0$ and $\phi(b) = 0$.
Also,
$\phi(a \vee b) = \phi(a) \vee \phi(b)$ ($\because \phi$ is join homomorphism)
$\phi(a \vee b) = 0 \vee 0$
$\phi(a \vee b) = 0$
$\Rightarrow$ ~~a~~ $a \vee b \in I$

Now, by using absorption property,
$a = a \vee (a \wedge b)$
$\Rightarrow \phi(a) = \phi[a \vee (a \wedge b)]$

$\Rightarrow 0 = \phi(a) \vee \phi(a \wedge b)$
$\Rightarrow 0 = 0 \vee \phi(a \wedge b)$
$\Rightarrow 0 = \phi(a \wedge b)$
$\Rightarrow a \wedge b \in I$

$\therefore I$ is sublattice of L.

Now consider $a \in I$ be any elements & $x \in L$ be any element.

$\phi(a) = 0$

Now, by absorption property,

$a = a \vee (a \wedge x)$

$\Rightarrow \phi(a) = \phi[a \vee (a \wedge x)]$

$\Rightarrow 0 = \phi(a) \vee \phi(a \wedge x)$

$\Rightarrow 0 = 0 \vee \phi(a \wedge x)$

$\Rightarrow 0 = \phi(a \wedge x)$

$\Rightarrow a \wedge x \in I$ $\quad \forall a \in I, \forall x \in L.$

$\therefore$ I is an ideal of L.

As I is onto join homomorphism.

$\Rightarrow$ I is proper ideal of L.

Subclaim:- I is prime ideal.

Let, $a, b \in L$ be any elements, such that, $a \wedge b \in I$

$\Rightarrow \phi(a \wedge b) = 0$

$\Rightarrow \phi(a) \wedge \phi(b) = 0$ $\quad$ ($\because \phi$ is meet homomorphism)

$\Rightarrow$ Either $\phi(a) = 0$ or $\phi(b) = 0$

$\Rightarrow$ Either $a \in I$ or $b \in I$

$\therefore$ I is prime ideal.

Note:-

1) If I is ideal of lattice L generated by 'a' denoted by $(a] = \{x \in L / x \leq a\}$ Then it's dual is denoted and defined by,

$D = [a) = \{x \in L \mid x \geq a\}$.

2] $D(L)$ is set of all duals of $I(L)$.

3] $D(L)$ is lattice under the conditions.
If $A, B \in D(L)$ then
i] $A \vee B = [A \cup B)$
ii] $A \wedge B = [A \cap B)$

4] $D_0 L = D(L) \cup \{\phi\}$

Convex subset:-

A ~~subset~~ non-empty subset C of lattice 'L' is said to be convex iff for $a, b \in C$ and $d \in L$ such that $a \leq d \leq b \Rightarrow d \in C$.

Th^m 6:-

Let I be an ideal and D be dual of I of a lattice L, if $I \cap D \neq \phi$ then, $I \cap D$ is convex sublattice of L, and every convex sublattice can be expressed in this form in one and only one way.

→ Proof:-

Let I be an ideal of lattice L.
Let D be dual of I.
Clearly. I and D are sublattices of L.

$\Rightarrow$ $I \cap D$ is sublattice of L.

Claim:- $I \cap D$ is convex.
As $I \cap D$ is non-empty subset of L.

i.e. $\phi \neq I \cap D \subseteq L$

Let, $a, b \in I \cap D$ be any elements.

Let, $c \in L$ be any element such that.

$a \le c \le b$.

As $b \in I$ and $c \le b$

$\Rightarrow c \in I$ $\quad$ ($\because$ I is ideal)

As $a \in D$ and $a \le c$

$\Rightarrow c \in D$ $\quad$ ($\because$ D is Ideal)

$\therefore c \in I \cap D$

$\therefore I \cap D$ is convex subset of L.

$\therefore I \cap D$ is convex sublattice of L.

Let, P be any convex sublattice of lattice L.

Let, $I = (P]$ and $D = [P)$

<u>Claim</u>:- $P = I \cap D$

As $I \cap D \neq \phi \subseteq L$

Let, $a \in I \cap D$

$\Rightarrow a \in I = (P]$

$\Rightarrow \cancel{a \not\le c}$ $\exists\, c \in (P]$ such that $a \le c$.

Simillarly,

As $a \in D = [P)$

$\Rightarrow \cancel{c \not\le a}$ $\exists\, d \in [P)$ such that $d \le a$.

$\therefore \exists\, c, d \in P$ such that $d \le a \le c$

$\Rightarrow a \in P$ $\quad$ ($\because$ P is convex sublattice).

$\therefore I \cap D \subseteq P$ —— ①

Clearly,

$$P \subseteq I \text{ and } P \subseteq D$$

$\Rightarrow \quad P \cap P \subseteq I \cap D$

$\Rightarrow \quad P \subseteq I \cap D \qquad \text{——— (2)}$

By (1) and (2),

$$P = I \cap D.$$

<u>Uniqueness</u>:-

Suppose $P = I \cap D = I_1 \cap D_1$

Since $P \subseteq I_1$ we have, $(P] = I_1$

$\Rightarrow I \subseteq I_1$

Let,

$a \in I_1$ and let ~~c be~~ $c \in P$ be an arbitrary elements.

$\Rightarrow \quad a \vee c \in I_1$

$\Rightarrow \quad a \vee c \geqslant c \in D_1$

$\Rightarrow \quad a \vee c \in D_1$

Thus,

$$a \vee c \in I_1 \cap D_1 = P$$

Finally, $a \leq a \vee c \in P$

$\Rightarrow \quad a \in (P] = I$

$\Rightarrow \quad I_1 \subseteq I$

$\therefore \quad I = I_1$ and consequently $D = D_1$.

Hence the uniqueness.

Congruence Relation :-

An equivalence relation θ on a lattice L is called congruence relation of L if for all $a_0, a_1, b_0, b_1 \in L$,

$a_0 \equiv b_0 (\theta)$ and $a_1 \equiv b_1 (\theta)$ then

$$\left.\begin{array}{l} 1]\ a_0 \wedge a_1 \equiv b_0 \wedge b_1 (\theta) \\ 2]\ a_0 \vee a_1 \equiv b_0 \vee b_1 (\theta) \end{array}\right\} \text{(substitution properties)}.$$

Note:-

1] We will use notations $a \theta b$ or $a \equiv b(\theta)$ or $a \overset{\theta}{\equiv} b$ have same meaning.

Congruence class:-

The equivalence classes under a congruence relation θ are called congruence classes.

The congruence class of $a \in L$ is denoted by $[a]$ or $[a]_\theta$ and defined by,

$[a] = \{ x \in L \mid x \equiv a(\theta) \}$.

Note:-

1] The set of all congruence classes for θ is denoted by $\frac{L}{\theta}$.

2] The set of all congruence relations on L is denoted by $\underline{\underline{con(L)}}$.

Th^m 7:-

Let θ be a congruence relation on lattice L then for every $a \in L$ $[a]_\theta$ is convex sublattice of L.

<u>Proof</u>:-

Let, $[a]_\theta = \{x \in L \mid x \equiv a(\theta)\}$

Clearly, $\phi \neq [a]_\theta \subseteq L$

Let,

$x, y \in [a]_\theta$ be any elements.

Then by substitution property,

$x \equiv a(\theta)$ and $y \equiv a(\theta)$

1] $x \wedge y \equiv a \wedge a(\theta)$

$\Rightarrow x \wedge y \equiv a(\theta)$

$\Rightarrow x \wedge y \in [a]_\theta \qquad \forall\, x, y \in [a]_\theta$. —— ①

2] $x \vee y \equiv a \vee a(\theta)$

$\Rightarrow x \vee y \equiv a(\theta)$

$\Rightarrow x \vee y \in [a]_\theta \qquad \forall\, x, y \in [a]_\theta$. —— ②

By ① + ②, $[a]_\theta$ is sublattice of L.

<u>Claim</u>:- $[a]_\theta$ is convex.

Let, $x, y \in [a]_\theta$ be any elements.

Consider, $x \le z \le y$ for some $z \in L$.

$\Rightarrow x \equiv a(\theta)$ and $y \equiv a(\theta)$

also $z \equiv z(\theta)$.

$\Rightarrow x = x \wedge z \equiv a \wedge z(\theta)$

$\Rightarrow x \wedge z \equiv a \wedge z(\theta) \equiv a(\theta) \qquad$ [$\because$ By transitivity

$\Rightarrow x \wedge z \equiv a(\theta)$

—— ①

Similarly,

$z \vee y \equiv a(\theta)$ — (ii)

$\Rightarrow x \wedge z \equiv z \vee y(\theta)$

$\Rightarrow (x \wedge z) \vee z \equiv (z \vee y) \vee z(\theta)$

$\Rightarrow z \equiv z \vee y(\theta)$

$\Rightarrow z \equiv a(\theta)$

$\Rightarrow z \in [a]_\theta$

$\therefore [a]_\theta$ is convex subset of L.

$\therefore [a]_\theta$ is convex sublattice of L.

VIMP
10-12 Marks

Th^m 8:-

A reflexive binary relation θ is congruence relation iff the following three properties satisfy for $x, y, z \in L$.

i] $x \equiv y(\theta)$ iff $x \wedge y \equiv x \vee y(\theta)$

ii] $x \le y \le z$, $x \equiv y(\theta)$, $y \equiv z(\theta)$ then $x \equiv z(\theta)$.

iii] $x \le y$, $x \equiv y(\theta) \Rightarrow x \wedge t \equiv y \wedge t(\theta)$

$x \vee t \equiv y \vee t(\theta)$, $t \in L$.

→ Proof:-

Given that θ is reflexive binary relation on lattice L.

Let, θ is congruence relation on lattice L.

I] Consider $x \equiv y(\theta)$ — (*)

Claim:- $x \wedge y \equiv x \vee y(\theta)$

As θ is congruence relation on lattice L we get $x \equiv x(\theta)$ and $y \equiv y(\theta)$

By using substitution property

$x \wedge y \equiv x \wedge y\ (\theta)$ and $x \wedge x \equiv y \wedge x\ (\theta)$

$\Rightarrow x \wedge y \equiv x \wedge y\ (\theta)$ and $x \equiv x \wedge y\ (\theta)$

$\Rightarrow x \equiv x \wedge y\ (\theta)$

— (I)

Again by using substitution property,

$x \vee y \equiv y \vee y\ (\theta)$

$\Rightarrow x \vee y \equiv y\ (\theta)$

— (II)

By using (I), (II) + (*),

$x \wedge y \equiv x \vee y\ (\theta)$

Conversely,

consider, $x \wedge y \equiv x \vee y\ (\theta)$ — (III)

Claim:- $x \equiv y\ (\theta)$

As θ is congruence relation we have,

$x \equiv x\ (\theta)$ and $y \equiv y\ (\theta)$

By using join substitution property we get,

$x \vee (x \wedge y) \equiv x \vee (x \vee y)\ (\theta)$

$\Rightarrow x \equiv x \vee y\ (\theta)$ — (IV)

Similarly,

$(x \wedge y) \vee y \equiv (x \vee y) \vee y\ (\theta)$

$\Rightarrow y \equiv x \vee y\ (\theta)$ — (V)

By

By (IV) + (V)

$x \equiv y\ (\theta)$

$\therefore x \equiv y\ (\theta)$ iff $x \wedge y \equiv x \vee y\ (\theta)$ $\forall x, y \in L$

2] Consider,

$x \le y \le z$, $x \equiv y(\Theta)$, $y \equiv z(\Theta)$

~~Az~~ ~~Θ~~

Claim:-

As Θ is congruence relation on L

$\Rightarrow$ Θ is ~~symmetric~~ transitive.

$\Rightarrow$ $x \equiv z(\Theta)$.

3] consider, $x \le y$, $x \equiv y(\Theta)$

Let $t \in L$ be any element.

As Θ is congruence relation on lattice L

we get,

$t \equiv t(\Theta)$.

By using join substitution we get,

$x \vee t \equiv y \vee t(\Theta)$ $\forall t \in L$

By using meet substitution we get,

$x \wedge t \equiv y \wedge t(\Theta)$ $\forall t \in L$

$\therefore$ $x \le y$, $x \equiv y(\Theta)$

$\Rightarrow$ $x \vee t \equiv y \vee t(\Theta)$

$x \wedge t \equiv y \wedge t(\Theta)$.

Conversely.

Suppose Θ satisfies three properties.

i] $x \equiv y(\Theta)$ iff $x \wedge y \equiv x \vee y(\Theta)$

ii] $x \le y \le z$, $x \equiv y(\Theta)$, $y \equiv z(\Theta)$ then $x \equiv z(\Theta)$

iii] $x \le y$, $x \equiv y(\Theta) \Rightarrow x \wedge t \equiv y \wedge t(\Theta)$

$x \vee t \equiv y \vee t(\Theta)$, $\forall t \in L$.

Claim:- Θ is congruence relation.

By hypothesis Θ is reflexive.

1] Symmetric :-

We have $x \equiv y(\Theta)$ iff $x \wedge y \equiv x \vee y(\Theta)$.

$\Leftrightarrow$ $x \wedge y \equiv x \vee y\,(\Theta)$

$\Leftrightarrow$ $y \wedge x \equiv y \vee x(\Theta)$

($\because$ $x, y \in L$ & L is lat commutative satisfy).

$\Leftrightarrow$ $y \equiv x(\Theta)$

$\therefore$ Θ is symmetric.

2] Transitivity :-

Let $x \equiv y(\Theta)$ and $y \equiv z(\Theta)$

$\Leftrightarrow$ $x \wedge y \equiv x \vee y(\Theta)$ and $y \wedge z \equiv y \vee z(\Theta)$ — (VI)

As $x \wedge y \leq x \vee y$

Take $t = y \vee z$ and apply third condition

$(x \wedge y) \vee t \equiv (x \vee y) \vee t(\Theta)$

$\Rightarrow$ $(x \wedge y) \vee (y \vee z) \equiv (x \vee y) \vee (y \vee z)(\Theta)$

$\Rightarrow$ $[(x \wedge y) \vee y] \vee z \equiv x \vee (y \vee y) \vee z(\Theta)$

$\Rightarrow$ $y \vee z \equiv x \vee y \vee z(\Theta)$ — (VII)

Take $t = x \wedge y$ and apply third condition.

As $y \wedge z \leq y \vee z$

$\Rightarrow$ $(y \wedge z) \wedge t \equiv (y \vee z) \wedge t(\Theta)$

$\Rightarrow$ $(y \wedge z) \wedge (x \wedge y) \equiv (y \vee z) \wedge (x \wedge y)(\Theta)$

$\Rightarrow$ $z \wedge (y \wedge y) \wedge x \equiv [(y \vee z) \wedge y] \wedge x(\Theta)$

($\because$ commutative & associative property)

$\Rightarrow$ $z \wedge y \wedge x \equiv y \wedge x(\Theta)$ — (VIII)

We have,

$x \wedge y \wedge z \leq x \wedge y \leq x \vee y$

From eq^n (VIII)

$x \wedge y \wedge z \equiv y \wedge x (\theta)$

From eq^n (VI)

$x \wedge y \equiv x \vee y (\theta)$

Then by using (ii) condition,

$x \wedge y \wedge z \equiv x \vee y (\theta)$ — (IX)

As. $x \wedge y \wedge z \leq x \vee y$

Take $t = y \wedge z$ and apply (iii) condition

$(x \wedge y \wedge z) \wedge t \equiv (x \vee y) \wedge t (\theta)$

$\Rightarrow (x \wedge y \wedge z) \wedge (y \wedge z) \equiv (x \vee y) \wedge (y \wedge z) (\theta)$

$\Rightarrow x \wedge [y \wedge z] \equiv [(x \vee y) \wedge y] \wedge z (\theta)$

$\Rightarrow x \wedge y \wedge z \equiv y \wedge z (\theta)$ — (X)

As $x \wedge y \wedge z \leq y \wedge z \leq y \vee z$

Then by (ii) condition, to eq^n (VII) & (X)

$x \wedge y \wedge z \equiv y \vee z (\theta).$ — (XI)

{As $x \wedge y \wedge z \leq y \vee z (\theta) \leq x \vee y \vee z$

and from eq^n (IX) & (XI) and apply (ii) condition we get,

$x \wedge y \wedge z \equiv x \vee y \vee z (\theta)$ — (XII)

We know that,

[Th^m :- If θ satisfies (iii) condition then for $a, b, c, d \in L$ with $b, c \in [a, d]$ and if $a \equiv d (\theta)$ then $b \equiv c (\theta)$.]

$\therefore$ We get $x, z \in [x \wedge y \wedge z, x \vee y \vee z]$

$\therefore \quad x \equiv z(\theta)$

Hence the transitivity.

3] Substitution properties:-

First we prove if $x \equiv y(\theta)$ then $x \vee t \equiv y \vee t(\theta)$

Let, $x \equiv y(\theta) \Leftrightarrow x \wedge y \equiv x \vee y(\theta)$

[$\because$ By condition (i)].

As $x \wedge y \leq x \vee y$ and $x \wedge y \equiv x \vee y(\theta)$ then by condition (iii),

$(x \wedge y) \vee t \equiv (x \vee y) \vee t(\theta)$

As $x \wedge y \leq x \leq x \vee y$

$\Rightarrow (x \wedge y) \vee t \leq x \vee t \leq (x \vee y) \vee t$

Similarly,

$(x \wedge y) \vee t \leq y \vee t \leq (x \vee y) \vee t$

$\Rightarrow x \vee t, y \vee t \in [(x \wedge y) \vee t, (x \vee y) \vee t]$

$\Rightarrow x \vee t \equiv y \vee t(\theta)$ ($\because$ By thm above in bracket XIII)

Let, $x_1 \equiv y_1(\theta)$ and $x_2 \equiv y_2(\theta)$

$\Rightarrow x_1 \vee x_2 \equiv y_1 \vee x_2(\theta)$ ($\because$ By XIII).

Similarly,

$\cancel{x_1 \wedge x_2 \equiv y_1 \wedge y_2(\theta)}$

$\cancel{y_1 \vee x_2}$ $x_2 \vee y_1 \equiv y_2 \vee y_1(\theta)$ ($\because$ By XIII)

$\Rightarrow y_1 \vee x_2 \equiv y_1 \vee y_2(\theta)$

Then by transitivity,

$x_1 \vee x_2 \equiv y_1 \vee y_2(\theta)$.

Simillarly we prove,

$$x_1 \wedge x_2 \equiv y_1 \wedge y_2 (\Theta)$$

$\therefore$ By ①, ② + ③, Θ is congruence relation on lattice L.

Quotient lattice / Factor lattice:-

Let L be a lattice and Θ be a congruence relation on L then $\frac{L}{\Theta} = \{[a]_\Theta / a \in L\}$ be a set of all congruence classes forms a lattice w.r.to

1) $[a]_\Theta \wedge [b]_\Theta = [a \wedge b]_\Theta$

2) $[a]_\Theta \vee [b]_\Theta = [a \vee b]_\Theta$ $\quad \forall a, b \in L$,

is called quotient lattice or factor lattice.

Q. With usual notations prove that $\frac{L}{\Theta}$ is lattice.

→ Let,

L be a lattice and Θ is congruence relation on L.

Then we know that,

$$\frac{L}{\Theta} = \{[a]_\Theta / a \in L\}$$

We define meet and join for $\frac{L}{\Theta}$ as,

1) $[a]_\Theta \wedge [b]_\Theta = [a \wedge b]_\Theta$

2) $[a]_\Theta \vee [b]_\Theta = [a \vee b]_\Theta$ $\quad \forall a, b \in L$

Claim:- $\frac{L}{\Theta}$ is lattice.

I] Idempotent property :-

Let $[b]_\theta \in \frac{L}{\theta}$ be any element.

consider,

$[b]_\theta \wedge [b]_\theta = [b \wedge b]_\theta = [b]_\theta$

$[b]_\theta \vee [b]_\theta = [b \vee b]_\theta = [b]_\theta$

$\therefore \frac{L}{\theta}$ satisfies idempotent property.

II] Commutative property :-

Let, $[a]_\theta$ and $[b]_\theta \in \frac{L}{\theta}$ be any element

i) consider,

$[a]_\theta \vee [b]_\theta = [a \vee b]_\theta$ ($\because$ By def^n)

$= [b \vee a]_\theta$ ($\because$ L is lattice)

$= [b]_\theta \vee [a]_\theta$ ($\because$ By def^n)

ii) consider,

$[a]_\theta \wedge [b]_\theta = [a \wedge b]_\theta$ ($\because$ By def^n)

$= [b \wedge a]_\theta$ ($\because$ L is lattice)

$= [b]_\theta \wedge [a]_\theta$ ($\because$ By def^n)

$\therefore$ By ① & ⑪ $\frac{L}{\theta}$ satisfies commutative properties.

III] Associative property :-

Let, $[a]_\theta, [b]_\theta, [c]_\theta \in \frac{L}{\theta}$ be any elements.

i) consider,

$[a]_\theta \vee \{[b]_\theta \vee [c]_\theta\} = [a]_\theta \vee [b \vee c]_\theta$ (By def^n)

$= [a \vee (b \vee c)]_\theta$ ($\because$ By def^n)

$= [(a \vee b) \vee c]_\theta$ ($\because$ L is lattice)

$$[a]_\theta \vee \{[b]_\theta \vee [c]_\theta\} = [a\vee b]_\theta \vee [c]_\theta \quad (\because \text{By def}^n)$$
$$= \{[a]_\theta \vee [b]_\theta\} \vee [c]_\theta \quad (\because \text{By def}^n)$$

ii] consider,

$$[a]_\theta \wedge \{[b]_\theta \wedge [c]_\theta\} = [a]_\theta \wedge [b\wedge c]_\theta$$
$$= [a\wedge (b\wedge c)]_\theta$$
$$= [(a\wedge b)\wedge c]_\theta \quad (\because L \text{ is latt}^e)$$
$$= [a\wedge b]_\theta \wedge [c]_\theta$$
$$= \{[a]_\theta \wedge [b]_\theta\} \wedge [c]_\theta$$

$\therefore$ By (i) & (ii) $\frac{L}{\theta}$ satisfies associative property

IV] Absorption property :-

Let, $[a]_\theta$ and $[b]_\theta \in \frac{L}{\theta}$ be any elements.

i] consider,

$$[a]_\theta \wedge \{[a]_\theta \vee [b]_\theta\} = [a]_\theta \wedge [a\vee b]_\theta \quad (\text{By def}^n)$$
$$= [a\wedge (a\vee b)]_\theta \quad (\text{By def}^n)$$
$$= [a]_\theta \quad (\because L \text{ is latt}^e)$$

ii] consider,

$$[a]_\theta \vee \{[a]_\theta \wedge [b]_\theta\} = [a]_\theta \vee [a\wedge b]_\theta \quad (\text{By def}^n)$$
$$= [a\vee (a\wedge b)]_\theta \quad (\text{By def}^n)$$
$$= [a]_\theta \quad (\because L \text{ is latt}^e)$$

$\therefore$ By (i) & (ii) $\frac{L}{\theta}$ satisfies absorption property

$\Rightarrow$ By (I) to (IV) $\frac{L}{\theta}$ is lattice.

Th^m 9:-

A congruence relation θ on lattice L satisfies $x \le y$, $x \equiv y(\theta)$ then / $\Rightarrow x \wedge t \equiv y \wedge t(\theta)$

$x \vee t \equiv y \vee t(\theta) \quad \forall t \in L$

then prove that $a,b,c,d \in L$ such that, $b,c \in [a,d]$ and $a \equiv d(\theta)$ then $b \equiv c(\theta)$.

→ Proof:-

Let $t = b \wedge c$.

As $a \le d$ and $a \equiv d(\theta)$

$$\left.\begin{aligned} \Rightarrow \quad & a \wedge t \equiv d \wedge t(\theta) \\ & a \vee t \equiv d \vee t(\theta) \end{aligned}\right\} \text{By hypothesis}$$

$$\Rightarrow a \vee (b \wedge c) \equiv d \vee (b \wedge c)(\theta) \quad \text{---} (1)$$

As $a \le b, c \le d$

$\Rightarrow \quad a \le b \wedge c$ and $b \wedge c \le d$

$\therefore$ Eq^n (1) becomes,

$$b \wedge c \equiv d(\theta) \quad \text{with} \quad b \wedge c \le d \quad \text{---} (2)$$

Take $t = b \vee c$ then from (2) we get,

$$t \wedge (b \wedge c) \equiv t \wedge d(\theta)$$

$$\Rightarrow (b \vee c) \wedge (b \wedge c) \equiv (b \vee c) \wedge d(\theta)$$

$$\Rightarrow \quad b \wedge c \equiv b \vee c(\theta)$$

$$\Rightarrow \quad b \wedge c \equiv b \vee c(\theta)$$

$$\Rightarrow \quad b \equiv c(\theta)$$

($\because \theta$ is congruence relation)

Th^m 10:- (Natural Homomorphism / Canonical Homomorphism)

The map $\psi_\theta : L \to \frac{L}{\theta}$ defined by

$\psi_\theta(x) = [x]_\theta \quad \forall x \in L$ is homomorphism of

L onto $\frac{L}{\Theta}$.

<u>Proof</u>:-

Let $a, b \in L$ be any elements.

i) $\psi_\Theta(a \wedge b) = [a \wedge b]_\Theta$ (∵ By def^n of ψ_Θ)

$= [a]_\Theta \wedge [b]_\Theta$ (∵ By def^n of $\frac{L}{\Theta}$)

$= \psi_\Theta(a) \wedge \psi_\Theta(b)$ (∵ By def^n of ψ_Θ)

ii) $\psi_\Theta(a \vee b) = [a \vee b]_\Theta$ (∵ By def^n of ψ_Θ)

$= [a]_\Theta \vee [b]_\Theta$ (∵ By def^n of $\frac{L}{\Theta}$)

$= \psi_\Theta(a) \vee \psi_\Theta(b)$ (∵ By def^n of ψ_Θ)

By ① + ⓘⓘ ψ_Θ is meet and join homomorphism.

$\Rightarrow$ ψ_Θ is homomorphism.

For any $[x]_\Theta \in \frac{L}{\Theta}$ $\exists$ $x \in L$ such that

$\psi_\Theta(x) = [x]_\Theta$

$\Rightarrow$ Pre-image of $[x]_\Theta$ is x.

$\Rightarrow$ ψ_Θ is onto.

∴ $\psi_\Theta : L \to \frac{L}{\Theta}$ defined by

$\psi_\Theta(x) = [x]_\Theta$ is always onto homomorphism.

Th^m 11 :-

Every homomorphic image of lattice L is Isomorphic to a suitable quotient lattice of L.

OR

Let L and L' be lattices if $\phi : L \to L'$ is a homomorphism and Θ is congruence relation on L defined by $x \equiv y(\Theta)$ iff $\phi(x) = \phi(y)$ then $\frac{L}{\Theta} \cong L'$.

→ Proof :-

Let $\phi : L \to L'$ is onto homomorphism. Note that, Θ is congruence relation on L. where $x \equiv y(\Theta)$ iff $\phi(x) = \phi(y)$. Since L' is homomorphic image of L under ϕ.

To prove $\frac{L}{\Theta} \cong L'$.

Define $\nu : \frac{L}{\Theta} \to L'$ def by

$$\nu\{[x]_\Theta\} = \phi(x)$$

i) To prove ν is well defined, and one-one.

Let, $[x]_\Theta = [y]_\Theta$

$$\Leftrightarrow \quad x \equiv y(\Theta)$$

$$\Leftrightarrow \quad \phi(x) = \phi(y)$$

$$\Leftrightarrow \quad \nu\{[x]_\Theta\} = \nu\{[y]_\Theta\}$$

$\therefore$ ν is well defined and one-one.

2] To prove ν is onto.

Let $\phi(x) \in L'$ be any element such that $\phi(x) = a$ $\quad$ [$\because \phi$ is onto].

$$\therefore \nu\{[x]_\theta\} = \phi(x) = a$$

$\therefore \nu$ is onto.

3] To prove ν is homomorphism.

Let $[x]_\theta$ and $[y]_\theta \in \frac{L}{\theta}$ be any elements

i] Consider,

$$\nu\{[x]_\theta \wedge [y]_\theta\} = \phi\,\nu\{[x \wedge y]_\theta\}$$
$$= \phi(x \wedge y)$$
$$= \phi(x) \wedge \phi(y)$$
$$\nu\{[x]_\theta \wedge [y]_\theta\} = \nu\{[x]_\theta\} \wedge \nu\{[y]_\theta\}$$

$\therefore \nu$ is meet homomorphism.

ii] Consider,

$$\nu\{[x]_\theta \vee [y]_\theta\} = \nu\{[x \vee y]_\theta\}$$
$$= \phi(x \vee y)$$
$$= \phi(x) \vee \phi(y)$$
$$\nu\{[x]_\theta \vee [y]_\theta\} = \nu\{[x]_\theta\} \vee \nu\{[y]_\theta\}$$

$\therefore \nu$ is join homomorphism.

$\therefore$ By (i) + (ii), ν is homomorphism.

By (1) to (3) ν is isomorphism.

$$\therefore \frac{L}{\theta} \cong L'.$$

Direct product of lattices :-

Let L_1 and L_2 be two lattices. $L_1 \times L_2$ be the cartesian product of L_1 and L_2 defined by.

$$L_1 \times L_2 = \{(l_1, l_2) \mid l_1 \in L_1, l_2 \in L_2\}$$

with meet and join defined as.

$$(a_1\ b_1) \wedge (a_2\ b_2) = (a_1 \wedge a_2 ,\ b_1 \wedge b_2)$$
$$(a_1\ b_1) \vee (a_2\ b_2) = (a_1 \vee a_2\ \ b_1 \vee b_2)$$

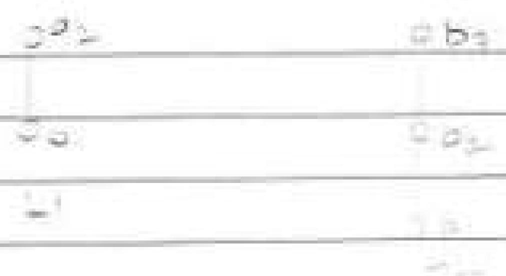
The covering relation is given by.

$(a_1\ b_1) \prec (a_2\ b_2)$ if $a_1 \prec a_2$, $b_1 = b_2$

<u>OR</u> $a_1 = a_2$, $b_1 \prec b_2$

Then $L_1 \times L_2$ is direct product of lattice L_1 and L_2.

Consider the lattices defined by



Find $L_1 \times L_2$ and $L_2 \times L_1$.

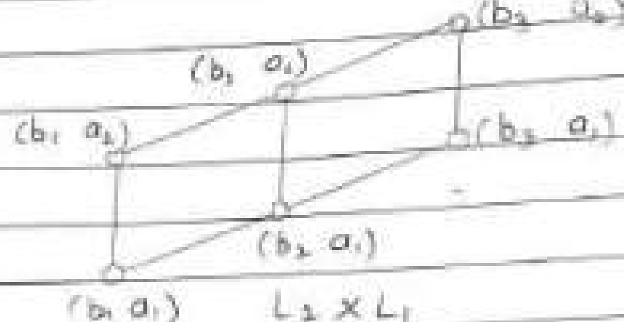
Let.

① Here $L_1 \times L_2 = \{(a_1\ b_1)\ (a_1\ b_2)\ (a_1\ b_3)$
$(a_2\ b_1)\ (a_2\ b_2)\ (a_2\ b_3)\}$

<u>$L_1 \times L_2$</u>

This is diagrammatic representation of $L_1 \times L_2$

② Here $L_2 \times L_1 = \{(b_1, a_1)\ (b_1\ a_2)\ (b_2\ a_1)\ (b_2\ a_2)$
$(b_3\ a_1)\ (b_3\ a_2)\}$



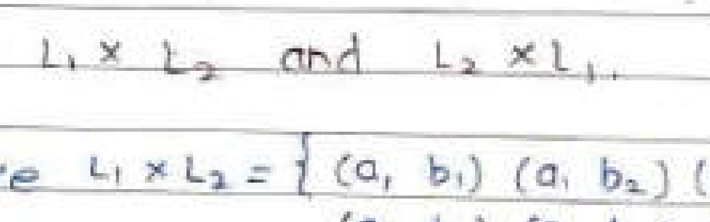
$L_2 \times L_1$

This is diagrammatic representation of $L_2 \times L_1$

Note:-
1] $L_1 \times L_2 \neq L_2 \times L_1$

2] $L_1 \times L_2 \cong L_2 \times L_1$

2] Consider the lattices defined by

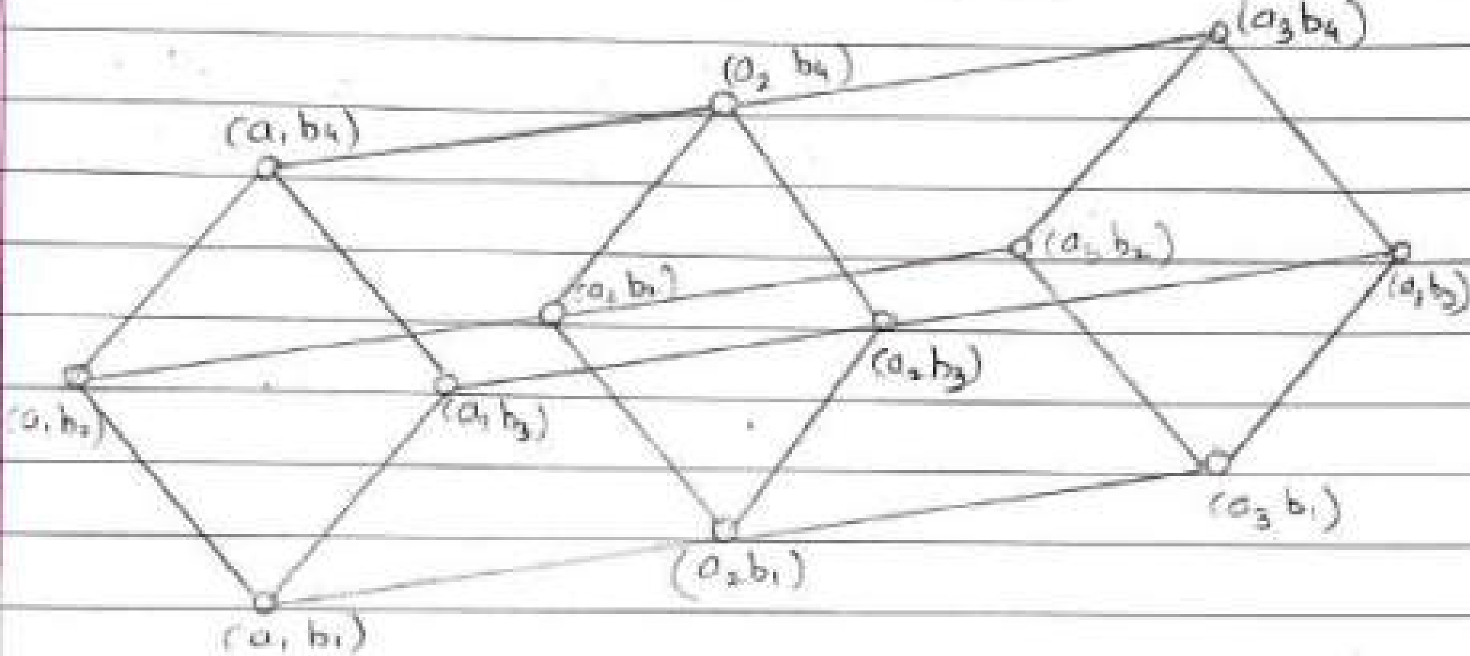


Find $L_1 \times L_2$ and $L_2 \times L_1$.

→ Let,

① Here $L_1 \times L_2 = \{(a_1, b_1)\ (a_1\ b_2)\ (a_1\ b_3)\ (a_1\ b_4)$
$(a_2\ b_1)\ (a_2\ b_2)\ (a_2\ b_3)\ (a_2\ b_4)$
$(a_3\ b_1)\ (a_3\ b_2)\ (a_3\ b_3)\ (a_3\ b_4)\}$

② Here $L_2 \times L_1 = \{(b_1\ a_1)\ (b_1\ a_2)\ (b_1\ a_3)$
$(b_2\ a_1)\ (b_2\ a_2)\ (b_2\ a_3)$
$(b_3\ a_1)\ (b_3\ a_2)\ (b_3\ a_3)$
$(b_4\ a_1)\ (b_4\ a_2)\ (b_4\ a_3)\}$



(a, b4)
(a2 b4)
(a3 b4)
(a, b2)
(a, b3)
(a2 b2)
(a2 b3)
(a3 b2)
(a3 b3)
(a, b1)
(a2 b1)
(a3 b1)

Th^m 12 :-

Let L and L' and K and K' be lattices such that $L \cong L'$ and $K \cong K'$ then $L \times L' \cong K \times K'$.

Proof :-

Let $\theta : L \to L'$ and $\phi : K \to K'$ be isomorphisms.

Let,

$\psi : L \times K \to L' \times K'$ defined by,

$$\psi(l \;\; k) = (\theta(l),\; \psi(k))$$

1) To prove ψ is well-defined and one-one.

Let,

$(l_1 \;\; k_1) = (l_2 \;\; k_2)$

$\Leftrightarrow \quad l_1 = l_2$ and $k_1 = k_2$

$\Leftrightarrow \quad \theta(l_1) = \theta(l_2)$ and $\phi(k_1) = \phi(k_2)$

$\Leftrightarrow \quad (\theta(l_1),\; \phi(k_1)) = (\theta(l_2),\; \phi(k_2))$.

$\Leftrightarrow \quad \psi(l_1 \;\; k_1) = \psi(l_2 \;\; k_2)$ $\quad \because$ By def^n

$\Leftrightarrow \quad \psi(l_1 \;\; k_1) = \psi(l_2 \;\; k_2)$ $\quad \forall\, l_1, l_2 \in L$, $k_1, k_2 \in K$

This shows that ψ is well-defined and one-one.

2) To prove ψ is onto.

Let, $(a, b) \in L' \times K'$ be any element.

$\Rightarrow$ a has preimage in L and b has preimage in K.

$\Rightarrow \exists\ x \in L$ such that $\theta(x) = a$ and
$\exists\ y \in K$ such that $\phi(y) = b$

$\therefore\ \psi(x\ \ y) = (\theta(x)\ ,\ \theta(y) = (a\ \ b)$
This shows that $(a\ \ b)$ has preimage $(x\ \ y) \in L \times K$.

$\Rightarrow$ ψ is onto.

3] To prove ψ is homomorphism.
a] ψ is meet homomorphism.
consider,

$$\psi[(l_1\ \ k_1) \wedge (l_2\ \ k_2)] = \psi[(l_1 \wedge l_2\ ,\ k_1 \wedge k_2)]$$

$$= [\theta(l_1 \wedge l_2)\ ,\ \phi(k_1 \wedge k_2)]$$

$$= [\theta(l_1) \wedge \theta(l_2)\ ,\ \phi(k_1) \wedge \phi(k_2)]$$

$$= [(\theta(l_1), \phi(k_1)) \wedge (\theta(l_2), \phi(k_2))]$$

$$\psi[(l_1\ \ k_1) \wedge (l_2\ \ k_2)] = \psi(l_1\ \ k_1) \wedge \psi(l_2\ \ k_2)$$

$\forall\ l_1, l_2 \in L$
$k_1, k_2 \in K$.

This shows that ψ is meet homomorphism

b] ψ is join homomorphism.
consider,

$$\psi[(l_1\ \ k_1) \vee (l_2\ \ k_2)] = \psi[(l_1 \vee l_2\ ,\ k_1 \vee k_2)]$$

$$= [\theta(l_1 \vee l_2),\ \phi(k_1 \vee k_2)]$$

$$\psi[(l_1, k_1) \vee (l_2\ k_2)] = [\theta(l_1) \vee \theta(l_2) \quad \phi(k_1) \vee \phi(k_2)]$$

$$= [(\theta(l_1), \phi(k_1)) \vee (\theta(l_2), \phi(k_2))]$$

$$\psi[(l_1, k_1) \vee (l_2\ k_2)] = \psi(l_1, k_1) \vee \psi(l_2\ k_2)$$

$\forall l_1, l_2 \in L$
$k_1, k_2 \in K$.

$\therefore$ By ⓐ & ⓑ
ψ is homomorphism.

$\therefore$ By ① to ③
ψ is isomorphism.
$\Rightarrow \quad L \times L' \cong K \times K'$

$$\psi[(l_1\ k_1)\vee(l_2\ k_2)] = [\theta(l_1)\vee\theta(l_2) \quad \phi(k_1)\vee\phi(k_2)]$$

$$= [(\theta(l_1),\phi(k_1)) \vee (\theta(l_2),\phi(k_2))]$$

$$\psi[(l_1\ k_1)\vee(l_2\ k_2)] = \psi(l_1\ k_1) \vee \psi(l_2\ k_2)$$

$\forall l_1, l_2 \in L$
$k_1, k_2 \in K$.

$\therefore$ By ⓐ + ⓑ
ψ is homomorphism.

$\therefore$ By ① to ③
ψ is isomorphism.
$\Rightarrow L \times L' \cong K \times K'$

Distributive lattice.

Distributive Lattice :-

A lattice L is called distributive lattice for any $x, y, z \in L$ we have.

i) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
ii) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Lemma 7:-

Consider the following identities and inequalities.

i) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
ii) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
iii) $(x \vee y) \wedge z \leq x \vee (y \wedge z)$.

Show that these are equivalent in any lattice L.

→ Proof:-

Let L be a lattice.
Let $x, y, z \in L$ be any elements.

i] **Claim** :- i] $\Rightarrow$ ii]

Let,

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ —— (I) $\forall\, x, y, z \in L$.

Consider, $a, b, c \in L$ be any elements

We choose,

$x = a \vee b$, $y = a$, $z = c$

Take LHS of eq^n (I),

$x \wedge (y \vee z) = (a \vee b) \wedge (a \vee c)$ —— (II)

Take RHS of eq^n (I),

$(x \wedge y) \vee (x \wedge z) = [(a \vee b) \wedge a] \vee [(a \vee b) \wedge c]$

$= a \vee [(a \vee b) \wedge c]$ ($\because$ Absorption)

$= a \vee [(a \wedge c) \vee (b \wedge c)]$ $\because$ By (I)

$= [a \vee (a \wedge c)] \vee (b \wedge c)$ ($\because$ Associative

$(x \wedge y) \vee (x \wedge z) = a \vee (b \wedge c)$ —— (III)

($\because$ Absorption)

By eq^n (I) to (III),

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ $\forall\, a, b, c \in L$

This shows that, i] $\Rightarrow$ ii].

2] **Claim** :- ii] $\Rightarrow$ i]

Let,

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ $\forall\, x, y, z \in L$ —— (I)

Consider, $a, b, c \in L$ be any elements.

We choose

$x = a \wedge b$, $y = a$, $z = c$.

Take LHS of eq^n (IV),
$$x \vee (y \wedge z) = (a \wedge b) \vee (a \wedge c) \quad \text{— (V)}$$

Take RHS of eq^n (IV),
$$(x \vee y) \wedge (x \vee z) = [(a \wedge b) \vee a] \wedge [(a \wedge b) \vee c]$$
$$= a \wedge [(a \wedge b) \vee c] \quad \because \text{(By Absorption)}$$
$$= a \wedge [(a \vee c) \wedge (b \vee c)] \quad \because \text{(By (II))}$$
$$= [a \wedge (a \vee c)] \wedge (b \vee c)] \quad \because \text{(Associati)}$$
$$(x \vee y) \wedge (x \vee z) = a \wedge (b \vee c) \quad \text{— (VI)} \quad \because \text{(Absorption)}$$

By eq^n (IV) to (VI),
$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$
This shows that ii) $\Rightarrow$ i)

3] Claim:- ii) $\Rightarrow$ iii)

Let,
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \quad \forall x, y, z \in L \quad \text{(VII)}$$
We know that,
$$z \leq x \vee z$$
$$\Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \quad \text{— By (VII)}$$
$$\geq (x \vee y) \wedge z$$
$$\Rightarrow (x \vee y) \wedge z \leq x \vee (y \wedge z).$$

This shows that ii) $\Rightarrow$ iii)

4] Claim:- iii) $\Rightarrow$ ii)

Let,

$(x \vee y) \wedge z \leq x \vee (y \wedge z) \quad \forall x, y, z \in L$ — (VIII)

Let, $a, b, c \in L$ be any elements.

We choose,

$x = a, \quad y = b$, and $z = a \vee c$ then by (VIII) we get,

$(a \vee b) \wedge (a \vee c) \leq a \vee [b \wedge (a \vee c)]$ — (IX)

We choose,

$x = a, \quad y = c$ and $z = b$ then by (VIII)

we get,

$(a \vee c) \wedge b \leq a \vee (c \wedge b)$

$\Rightarrow a \vee [(a \vee c) \wedge b] \leq a \vee [a \vee (c \wedge b)]$

$= (a \vee a) \vee (c \wedge b)$

$= a \vee (c \wedge b)$

$\Rightarrow a \vee [(a \vee c) \wedge b] \leq a \vee (c \wedge b)$. — (X)

By using (IX) + (X),

$(a \vee b) \wedge (a \vee c) \leq a \vee (c \wedge b)$. — (XI)

Similarly we prove,

$(a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$ — (XII)

By (XI) + (XII)

$(a \vee b) \wedge (a \vee c) = a \vee (b \wedge c)$

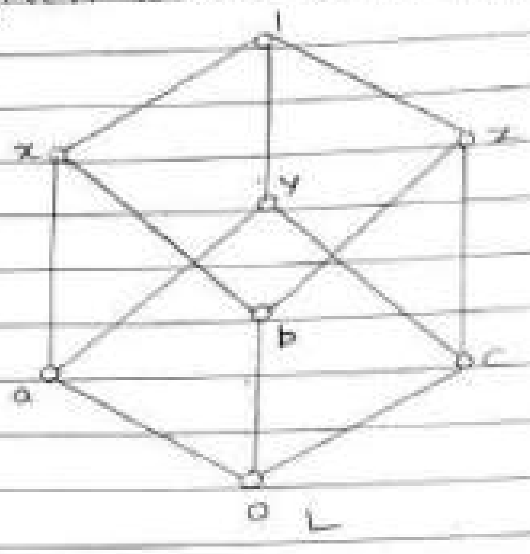
This shows that iii) $\Rightarrow$ ii).

By (1) to (4),

i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iii).

Hence the proof.

Q. Check given lattice is distributive or not.



→ Let,

1] Let, $x, y, z \in L$

Here, $x \vee (y \wedge z) = x \vee c = 1$

Similarly,

$(x \vee y) \wedge (x \vee z) = 1 \wedge 1 = 1$

$\therefore \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

2] Let, $x, b, z \in L$

Here, $x \vee (b \wedge z) = x \vee 0 = x$

Similarly,

$(x \vee b) \wedge (x \vee z) = x \wedge 1 = x$

$\therefore \quad x \vee (b \wedge z) = (x \vee b) \wedge (x \vee z)$

3] Let, $y, b, c \in L$

Here, $y \vee (b \wedge c) = y \vee 0 = y$

Similarly,

$(y \vee b) \wedge (y \vee c) = 1 \wedge y = y$

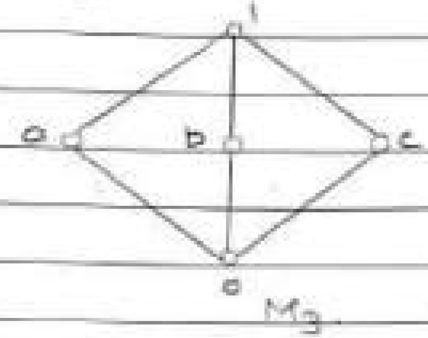
$\therefore \quad y \vee (b \wedge c) = (y \vee b) \wedge (y \vee c)$.

and so on.

$\therefore \forall\, l, m, n \in L, \; l \vee (m \wedge n) = (l \vee m) \wedge (l \vee n)$

$\therefore$ Given lattice is distributive.

Q. Check the given lattice is distributive or not.



→ Let,

i) Let $a, b, c \in M_3$

Here $a \vee (b \wedge c) = a \vee 0 = a$

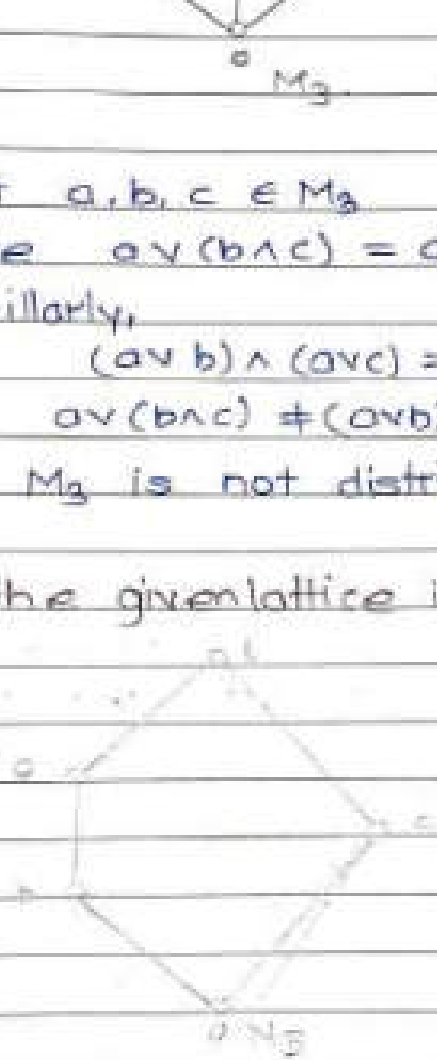
Similarly,

$(a \vee b) \wedge (a \vee c) = 1 \wedge 1 = 1$

$\therefore \quad a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$

$\therefore$ M_3 is not distributive lattice.

Q. Check the given lattice is distributive or not.



→ Let,

i) Let, $a, b, c \in N_5$

Here, $a \vee (b \wedge c) = a \vee 0 = a$

$(a \vee b) \wedge (a \vee c) = a \wedge 1 = a$

$\therefore \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

2] Let, $a, c, 1 \in N_5$

Here, $a \vee (c \wedge 1) = a \vee c = 1$

Similarly,

$(a \vee c) \wedge (a \vee 1) = 1 \wedge 1 = 1$

$\therefore \quad a \vee (c \wedge 1) = (a \vee c) \wedge (a \vee 1)$

3] Let, $b, a, c \in N_5$

Here, $b \vee (a \wedge c) = b \vee 0 = b$

and $(b \vee a) \wedge (b \vee c) = a \wedge 1 = a$

$\therefore \quad b \vee (a \wedge c) \neq (b \vee a) \wedge (b \vee c)$

$\therefore$ N_5 is not distributive lattice.

Note:-

1] Dual of distributive lattice is distributive.

Th^m 13:-

Let $x, y, z \in L$ be any elements

Show that identity $(x \wedge y) \vee (x \wedge z) = x \wedge [y \vee (x \wedge z)]$ is equivalent to $x \geq z \Rightarrow (x \wedge y) \vee z = x \wedge (y \vee z)$.

→ Proof:-

Let $x, y, z \in L$ be any elements.

Let $(x \wedge y) \vee (x \wedge z) = x \wedge [y \vee (x \wedge z)]$ —①

To prove, $x \geq z \Rightarrow (x \wedge y) \vee z = x \wedge (y \vee z)$.

Let, $x \geq z$

$\Rightarrow \quad x \wedge z = z$ and $x \vee z = x$

$\therefore$ Eq^n ① becomes,

$(x \wedge y) \vee z = x \wedge [y \vee z]$

$\therefore \quad (x \wedge y) \vee z = x \wedge (y \vee z)$

Conversely,

Let $x \geq z \Rightarrow (x \wedge y) \vee z = x \wedge (y \vee z)$ — (II)

We know that,

$x \geq x \wedge z$ and $z \geq x \wedge z$

As

$x \geq z$ then by (II)

$(x \wedge y) \vee x \wedge z = x \wedge [y \vee (x \wedge z)]$

Hence the proof.

Modular lattice :-

A lattice L is called modular if it satisfies for $x \geq z \Rightarrow (x \wedge y) \vee z = x \wedge (y \vee z)$ $\forall x, y, z \in L$.

Examples:-

1) Every chain is modular lattice.

2) Power set of any set form a modular lattice.

3) M_3 is modular lattice.

ie.

is modular lattice.

Let, $a, b, 1 \in M_3$ with $1 \geq a$

$\Rightarrow (1 \wedge b) \vee a = b \vee a = 1$

Also,

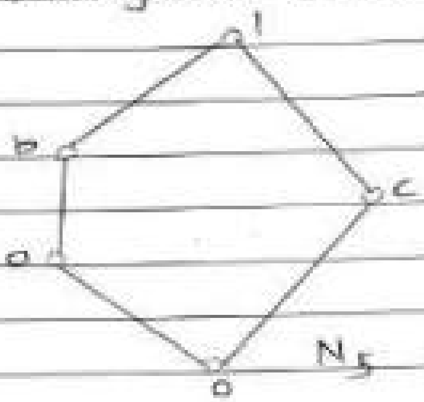
$1 \wedge (b \vee a) = 1 \wedge 1 = 1$

$\therefore 1 \geq a \Rightarrow (1 \wedge b) \vee a = 1 \wedge (b \vee a)$.

and so on.

$\therefore M_3$ is modular lattice.

Q. Check the given lattice is modular or not.



N_5

→ Let,

i) Let $a, b, c \in N_5$

Here $b \geq a$,

Now, $(b \wedge c) \vee a = 0 \vee a = a$

and $b \wedge (c \vee a) = b \wedge 1 = b$

$\therefore b \geq a \not\Rightarrow (b \wedge c) \vee a \neq b \wedge (c \vee a)$

$\therefore N_5$ is not modular lattice.

Note:-

1) N_5 is smallest non-modular lattice.

Median:-

Let, $a, b, c \in L$ be any elements if

$(a \wedge b) \vee (a \wedge c) \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \wedge (b \vee c)$ — ①

then the value of eq^n ① is called median of a, b, c.

Th^m 14:-

A lattice L is distributive lattice iff $\exists$ median $\forall a, b, c \in L$.

→ Proof:-

Let L is distributive lattice.

Let, $a, b, c \in L$ be any elements.

Claim:- Median of a, b, c exists.

Consider,

$(a \wedge b) \vee (a \wedge c) \vee (b \wedge c)$

$= (a \wedge b) \vee [(a \wedge c) \vee (b \wedge c)]$

$= (a \wedge b) \vee [(a \vee b) \wedge c]$ $\quad \because$ (L is distributive lattice).

$= (a \wedge b) \vee [c \wedge (a \vee b)]$ $\quad \because$ (commutativity in lattice).

$= [(a \wedge b) \vee c] \wedge [(a \wedge b) \vee (a \vee b)]$ $\quad \because$ (L is distri.).

$= [(a \wedge b) \vee c] \wedge [a \vee b]$. $\quad \because (a \wedge b \leq a \vee b)$.

$= (a \vee c) \wedge (b \vee c) \wedge (a \vee b)$

$= (a \vee b) \wedge (a \vee c) \wedge (b \vee c)$ $\quad \because$ (commutativity in L).

$\therefore (a \wedge b) \vee (a \wedge c) \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \wedge (b \vee c)$, $\forall a, b, c \in L$.

$\therefore \exists$ median $\forall a, b, c \in L$,

Conversely,

Let, $a, b, c \in L$ be any elements. and L has median $\forall a, b, c \in L$.

Claim:- L is distributive lattice.

First we show L is modular lattice,

Let $a \geq c$ and $(a \wedge b)$

$(a \wedge b) \vee (a \wedge c) \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \wedge (b \vee c)$ —①

As,

$a \geq c \Rightarrow a \wedge c = c$ and $a \vee c = a$

$\therefore$ Eq^n ① becomes.

$(a \wedge b) \vee c \vee (b \wedge c) = (a \vee b) \wedge a \wedge (b \vee c)$.

$\Rightarrow (a \wedge b) \vee [c \vee (b \wedge c)] = [(a \vee b) \wedge a] \wedge (b \vee c)$

$\Rightarrow (a \wedge b) \vee c = a \wedge (b \vee c)$

$\therefore a \geq c \Rightarrow (a \wedge b) \vee c = a \wedge (b \vee c)$ —②

$\Rightarrow$ L is modular lattice.

Now we take meet of element a with eq^n ①,

$\therefore a \wedge \{(a \wedge b) \vee (a \wedge c) \vee (b \wedge c)\}$

$= a \wedge \{(a \vee b) \wedge (a \vee c) \wedge (b \vee c)\}$ —③

Consider,

Let, $a \wedge \{(a \wedge b) \vee (a \wedge c) \vee (b \wedge c)\}$

$= a \wedge \{(a \wedge c) \vee (b \wedge c) \vee (a \wedge b)\}$ $\because$ [Commutative

$= \{a \wedge [(a \wedge c) \vee (b \wedge c)]\} \vee (a \wedge b)$ $\because$ [Modular lattice]

$= \{a \wedge [(b \wedge c) \vee (a \wedge c)]\} \vee (a \wedge b)$

$= \{[a \wedge (b \wedge c)] \vee (a \wedge c)\} \vee (a \wedge b)$ $\because$ [Modular lattice]

$a \wedge (b \wedge c)$
$= a \wedge (c \wedge b)$
$= (a \wedge c) \wedge b$

$= \{[(a \wedge c) \wedge b] \vee (a \wedge c)\} \vee (a \wedge b)$ [$\because$ Commutative & Associative]

$= (a \wedge c) \vee (a \wedge b)$ [Absorption property]

$a \wedge \{(a \wedge b) \vee (a \wedge c) \vee (b \wedge c)\} = (a \wedge c) \vee (a \wedge b)$

Similarly,

Let, $a \wedge \{(a \vee b) \wedge (a \vee c) \wedge (b \vee c)\}$

$= [a \wedge (a \vee b)] \wedge (a \vee c) \wedge (b \vee c)$ [$\because$ Modular lattice]

$= a \wedge [(a \vee c) \wedge (b \vee c)]$

$= [a \wedge (a \vee c)] \wedge (b \vee c)$ [$\because$ Associative]

$= a \wedge (b \vee c)$ [$\because$ Absorption]

$\therefore a \wedge \{(a \vee b) \wedge (a \vee c) \wedge (b \vee c)\} = a \wedge (b \vee c)$

$\therefore$ Eq^n ③ becomes,

$(a \wedge c) \vee (a \wedge b) = a \wedge (b \vee c)$

$(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c) \quad \forall a, b, c \in L.$

$\therefore$ L is distributive lattice.

Note:-

1) Prove that every distributive lattice is modular but converse need not be true.

→ Proof:-

Let L be any distributive lattice.

$\Rightarrow \exists$ median $\forall a, b, c \in L$

$\Rightarrow$ L is modular lattice.

For converse, take

Take $L = M_3$,

which is modular but not distributive.

Take L = Any chain

which is modular and distributive.

$\therefore$ Converse may or may not be true.

Bounded Poset:-

A poset is said to be bounded if it has both zero and unit elements.

Complement :-

Let L be a bounded lattice. Let $a \in L$ be any element ~~th~~ if $\exists$ $b \in L$ such that,

i) $a \wedge b = 0$

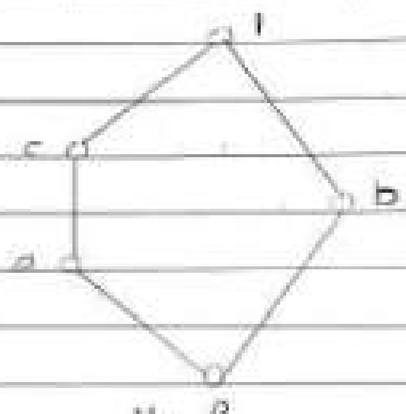
ii) $a \vee b = 1$,

then b is called complement of a.

i.e. $\Rightarrow$ complement of $a = b$.

$\Rightarrow$ $a' = b$.

1) Consider the lattice,



N_5

Find complement of 'b'.

$\rightarrow$ Let,

observe that,

$b \wedge a = 0$ and $b \vee a = 1$

$\Rightarrow$ $b' = a$

Also, $b \wedge c = 0$ and $b \vee c = 1$

$\Rightarrow$ $b' = c$.

Note:-

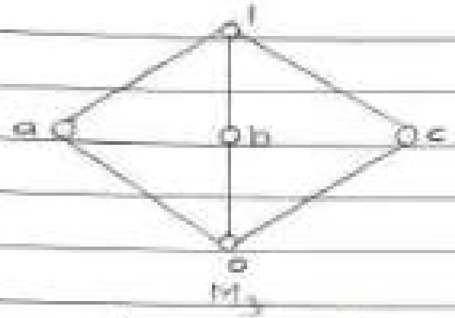
1) Complement of $a \in L$ is denoted by a'.

2) Complement of 0 is 1 and conversely.

2] Find complement of all elements of M_3 if exist.

→ Let,

We know that structure of M_3.



M_3

Clearly, $1' = 0$, $0' = 1$, $(a)' = b$ and c

$(b)' = a$ and c

$(c)' = a$ and b.

Complemented lattice :-

A lattice L is said to be complemented lattice if every element of L has complement.

e.g.

1] N_5 and M_3 are complemented lattices.

2] Consider the chain,

Clearly, a and b complements.

∴ L is not complemented lattice.

Note:-

1] A chain with atleast three elements is not complemented lattice.

distributive lattice example.

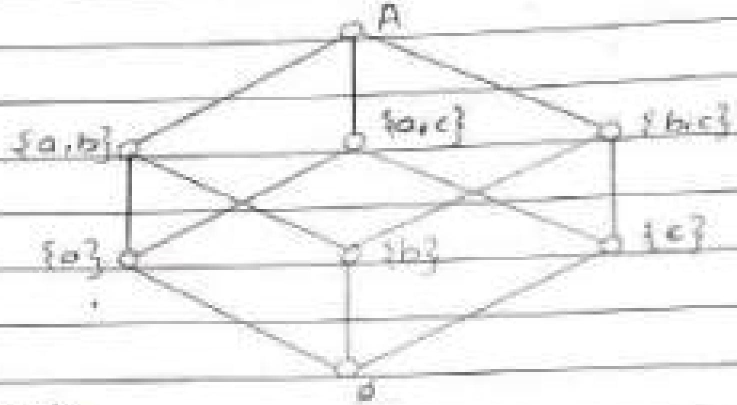
2] Let A be any set then power set of A i.e. P(A) is complemented lattice. Infact every element in this lattice have unique complement.

e.g.

For instance consider $A = \{a, b, c\}$

$\therefore P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, A\}$

Hasse Diagram of P(A).



Clearly,

$\phi' = A$

$A' = \phi$

$\{a\}' = \{b, c\}$ $\{a, b\}' = \{c\}$

$\{b\}' = \{a, c\}$ $\{a, c\}' = \{b\}$

$\{c\}' = \{a, b\}$ $\{b, c\}' = \{a\}$

All elements of the lattice P(A) have unique complements.

Th^m 15 :-

In a bounded distributive lattice an element can have only one complement.

OR

In a bounded distributive lattice an element can have unique complement.

→ Proof :-

Let L be a bounded distributive lattice

Let $a \in L$ such that complement of 'a' exists.

Let if possible b, c be two complements of a.

$\therefore \quad a \wedge b = 0 \quad$ and $\quad a \vee b = 1$
$\quad\quad a \wedge c = 0 \quad$ and $\quad a \vee c = 1$ $\Big\}$ — (I)

Consider,

$b = b \wedge 1$ $\quad (\because b \leq 1)$
$= b \wedge (a \vee c)$ $\quad (\because$ By (I)$)$
$= (b \wedge a) \vee (b \wedge c)$ $\quad (\because$ Distributive$)$
$= 0 \vee (b \wedge c)$
$= b \wedge c$

$\therefore \quad b \leq c$ — (II)

Consider,

$c = c \wedge 1$ $\quad (\because c \leq 1)$
$= c \wedge (a \vee b)$ $\quad (\because$ By (I)$)$
$= (c \wedge a) \vee (c \wedge b)$
$= 0 \vee (c \wedge b)$
$= c \wedge b$

$\therefore \quad c \leq b$ — (III)

$\therefore$ By (II) & (III)

$b = c$.

This shows that a have unique complement. Hence the proof.

De-Morgan's Identities :-

Th^m 16 :-

In a bounded distributive lattice L if a and b have complements a' and b' resp. Then $a \wedge b$ and $a \vee b$ have complements $a' \vee b'$ and $a' \wedge b'$ resp.

i.e. $(a \wedge b)' = a' \vee b'$ and $(a \vee b)' = a' \wedge b'$.

→ Proof:-

Let L be a bounded distributive lattice. Let $a, b \in L$ be any elements and complements of a and b exists.

Let,

$\therefore \; a \wedge a' = 0$ and $a \vee a' = 1$

$b \wedge b' = 0$ and $b \vee b' = 1$

Claim:- $(a \wedge b)' = a' \vee b'$

consider,

$(a \wedge b) \wedge (a' \vee b')$

$= [(a \wedge b) \wedge a'] \vee [(a \wedge b) \wedge b']$ ($\because$ Distributive property)

$= [a' \wedge (a \wedge b)] \vee [(a \wedge b) \wedge b']$ ($\because$ Commutative property)

$= [(a' \wedge a) \wedge b] \vee [a \wedge (b \wedge b')]$ ($\because$ Associative property)

$= [0 \wedge b] \vee [a \wedge 0]$

$= (0 \wedge b) \vee (a \wedge 0)$

$= 0 \vee 0$

$= 0$

$\therefore (a \wedge b) \wedge (a' \vee b') = 0$

Similarly,

Consider,

$(a \wedge b) \vee (a' \vee b')$

$= [a \vee (a' \vee b')] \wedge [b \vee (a' \vee b')]$ ($\because$ Distributive property)

$= [(a \vee a') \vee b'] \wedge [(b \vee b') \vee a']$ ($\because$ commutative & Associative)

$= (1 \vee b') \wedge (1 \vee a')$

$= 1 \wedge 1$

$= 1$

$\therefore (a \wedge b) \vee (a' \vee b') = 1$

$\therefore$ complement of $a \wedge b$ is $a' \vee b'$

$\Rightarrow \quad (a \wedge b)' = a' \vee b'$

2] Claim :- $(a \vee b)' = a' \wedge b'$

Consider,

$(a \vee b) \wedge (a' \wedge b')$

$= [a \wedge (a' \wedge b')] \vee [b \wedge (a' \wedge b')]$ ($\because$ Distributive property)

$= [(a \wedge a') \wedge b'] \vee [(b \wedge b') \wedge a']$ ($\because$ Commutative Associative)

$= [0 \wedge b'] \vee [0 \wedge a']$

$= (0 \wedge b') \vee (0 \wedge a')$

$= 0 \vee 0$

$= 0$

$\therefore (a \vee b) \wedge (a' \wedge b') = 0$

Similllarly,

Consider,

$(a \vee b) \vee (a' \wedge b')$

$= [(a \vee b) \vee a'] \wedge [(a \vee b) \vee b']$ ($\because$ Distributive property

$= [a' \vee (a \vee b)] \wedge [(a \vee b) \vee b']$ ($\because$ Commutative property)

$= [(a' \vee a) \vee b] \wedge [a \vee (b \vee b')]$ ($\because$ Associative property)

$= (1 \vee b) \wedge (a \vee 1)$

$= 1 \wedge 1$

$= 1$

$\therefore (a \vee b) \vee (a' \wedge b') = 1$

$\therefore$ Complement of $a \vee b$ is $a' \wedge b'$.

$\Rightarrow \quad (a \vee b)' = a' \wedge b'$

Hence the proof.

Thm 17:-

Prove that collection of all complements in distributive lattice 'L' forms a sublattice.

→ Proof:-

Let L be a distributive lattice.

Let H be a collection of all complements in L.

i.e. $H = \{a \in L \mid a' \text{ exists}\}$

Claim:- H is sublattice of L.

Clearly, $\phi \neq H \subseteq L$

Let, $a, b \in H$ be any elements.

$\Rightarrow$ a' and b' exists.

By De-Morgan's identities we have,

$(a \wedge b)' = a' \vee b'$

$(a \vee b)' = a' \wedge b'$

But we have,

$a', b' \in L$

$\Rightarrow a' \wedge b' \in L$ and $a' \vee b' \in L$

$\Rightarrow (a \vee b)' \in L$ and $(a \wedge b)' \in L$

$\Rightarrow$ ~~a∨b~~ complements of $a \vee b$ and $a \wedge b$ exists.

$\Rightarrow a \vee b \in H$ and $a \wedge b \in H \quad \forall a, b \in H$

This shows that H is sublattice of L.

Hence the proof.

Th^m 18 :-

In any distributive lattice $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ together implies $x = y$.

→ Proof:-

Let L be a distributive lattice.

Given that,

$$\left.\begin{array}{l} a \wedge x = a \wedge y \\ a \vee x = a \vee y \end{array}\right\} \text{— (I)} \quad \text{for } a, x, y \in L$$

We know that,

$$\begin{aligned}
x &= \cancel{a}\; x \wedge (a \vee x) && (\because \text{Absorption property}) \\
&= x \wedge (a \vee y) && (\because \text{By (I)}) \\
&= (x \wedge a) \vee (x \wedge y) && (\because L \text{ is distributive}) \\
&= (a \wedge x) \vee (x \wedge y) && (\because \text{commutative}) \\
&= (y \wedge a) \vee (x \wedge y) && (\text{By (I)}) \\
&= (y \wedge a) \vee (y \wedge x) && (\because \text{commutative}) \\
&= y \wedge (a \vee x) && (\because \text{Distributive}) \\
&= y \wedge (a \vee y) && (\because \text{By (I)}) \\
\therefore x &= y && (\text{Absorption property})
\end{aligned}$$

Hence the proof.

Th^m 19 :-

In a modular lattice if $x \geq y$ and $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ together implies $x = y$.

→ Proof:-

Let L be a modular lattice.

Given that,

$$x \geq y \text{ with } \left.\begin{array}{l} a \wedge x = a \wedge y \\ a \vee x = a \vee y \end{array} \quad \text{for } a, x, y \in L \right\} \text{— (I)}$$

We know that,

$x = x \wedge (a \vee z)$ (∵ Absorption)

$= x \wedge (y \vee a)$ (∵ By ①)

$= (x \wedge a) \vee y$ (∵ L is Modular)

$= (y \wedge a) \vee y$ (∵ By ②)

$\therefore x = y$ (∵ Absorption property).

Hence the proof.

Q. Prove that the collection of all subgroups of any group G forms a lattice.

→ Let,

Consider G be any group.

Let L(G) is the set of all subgroups of G.

i.e. $L(G) = \{H \mid H \text{ is subgroup of } G\}$

<u>Claim</u>:- L(G) is lattice.

Clearly, L(G) is non-empty.

Let, $H, K \in L(G)$ be any subgroups of G.

Define,

$$H \wedge K = H \cap K$$

We know that, intersection of two subgroups is again subgroup.

$\Rightarrow$ $H \wedge K$ is subgroup of G.

$\Rightarrow$ $H \wedge K \in L(G)$. $\forall H, K \in L(G)$.

Now we define,

$H \vee K$ = smallest subgroup of G properly containing HK

Clearly, $H \vee K$ is subgroup of G.

$\Rightarrow H \vee K \in L(G)$ $\forall H, K \in L(G)$

i.e. $\sup\{H, K\}$ and $\inf\{H, K\}$ belongs to $L(G)$.

$\therefore$ $L(G)$ forms a lattice.

1) Let $G = \mathbb{Z}_6$ under binary operation $\oplus_6$. Prove that $L(\mathbb{Z}_6)$ is lattice.

$\rightarrow$ Let,

We know that,

$\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ w.r.to $\oplus_6$

H_i clearly,

$H_1 = \{0\}$

$H_2 = \{0, 3\}$

$H_3 = \{0, 2, 4\}$

$H_4 = \mathbb{Z}_6$

are all subgroups $\mathbb{Z}_6$.

Here,

Here,

$H_2 \vee H_3 = \mathbb{Z}_6$

2) Let $G = S_3$ symmetric group on three letters prove that $L(S_3)$ is lattice.

$\rightarrow$ Let,

We know that,

$S_3 = \{\rho_0, \rho_1, \rho_2, \mu_1, \mu_2, \mu_3\}$

where, $\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $\mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $\mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

$\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ $\mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

where, $H_1 = \{\rho_0\}$
$H_2 = \{\rho_0, \mu_1\}$
$H_3 = \{\rho_0, \mu_2\}$
$H_4 = \{\rho_0, \mu_3\}$
$H_5 = \{\rho_0, \rho_1, \rho_2\}$
$H_6 = S_3$

are all subgroups of S_3.

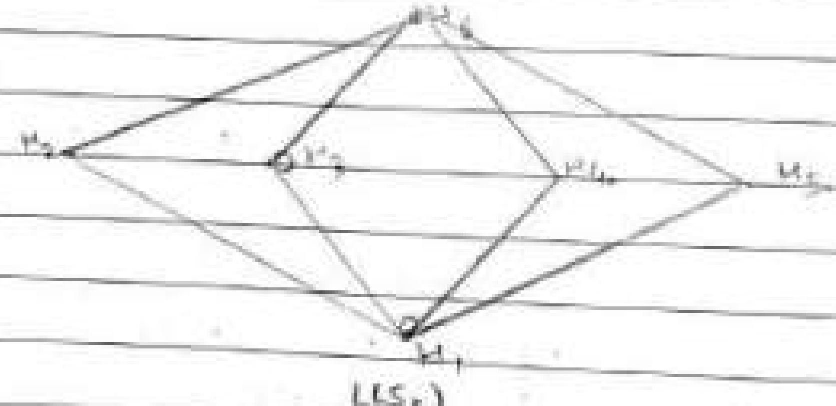
$\therefore L(S_3) = \{H_1, H_2, H_3, H_4, H_5, H_6\}$

Clearly, $H_1 \leq H_i$, $i = 1, 2, \dots 6$

$H_i \leq H_6$, $i = 1, 2, \dots 6$

and $\{H_2, H_3, H_4, H_5\}$ is antichain.

$\therefore$ Hasse diagram of $L(S_3)$ is,



$L(S_3)$

Here,

$H_2 \vee H_3 = \{$ smallest subgroup containing $H_2 H_3 \}$

complex.

Now,

$H_2 H_3 = \{ab \mid a \in H_2 \;\; b \in H_3\}$

$= \{\rho_0\rho_0,\ \rho_0\mu_2,\ \mu_1\rho_0,\ \mu_1\mu_2\}$

$= \{\rho_0\rho_0,\ \mu_2,\ \mu_1,\ \rho_1\} \subseteq S_3$

$\therefore \quad H_2 \vee H_3 = S_3$

Simillarly,

$H_4 \vee H_5$ = smallest subgroup containing H_4H_5.

$H_4H_5 = \{ab \mid a \in H_4,\ b \in H_5\}$

$= \{\rho_0\rho_0,\ \rho_0\rho_1,\ \rho_0\rho_2,\ \mu_3\rho_0,\ \mu_3\rho_1,\ \mu_3\rho_2\}$

$= \{\rho_0,\ \rho_1,\ \rho_2,\ \mu_3,\ \mu_1,\ \mu_2\} \leq S_3$

$H_4H_5 = S_3$

$\therefore \quad H_4 \vee H_5 = S_3$

Q. Prove that the collection of all normal subgroups of any group G forms an modular lattice.

→ Proof:-

Let G be any group.

Let M(G) = Collection of all normal subgroups of G.

i.e $M(G) = \{H \mid H \trianglelefteq G\}$.

Clearly, M(G) is lattice.

Infact M(G) is sublattice of L(G).

Claim:- M(G) is modular.

Let, $H, K, N \in M(G)$ be any elements

Let, $H \trianglelefteq N$ then we prove,

$(N \wedge K) \vee H = N \wedge (K \vee H)$

ie. $(N \cap K) \vee H = N \cap (K \vee H)$

ie. $(N \cap K) H = N \cap (KH)$

i.e $H(N\cap K) = (HK)\cap N$

As, $H \trianglelefteq N$

$\Rightarrow H \trianglelefteq HK$ and $H \leq N$

$\Rightarrow H\cap H \trianglelefteq (HK)\cap N$

$\Rightarrow H \trianglelefteq (HK)\cap N$

$\Rightarrow H \leq (HK)\cap N$ —— ①

Similarly,

$K\cap N \subseteq K \subseteq HK$ and $N\cap K \leq N$ i.e. $K\cap N \subseteq N$

$\Rightarrow (K\cap N)\cap(K\cap N) \subseteq (HK)\cap N$

$\Rightarrow K\cap N \subseteq (HK)\cap N$ —— ②

Intersection and complex of normal subgroups is again normal subgroup.

$\therefore H(K\cap N) \subseteq HK\cap N$ —— ③

Now, we will prove,

$HK\cap N \subseteq H(K\cap N)$.

Let, $x \in HK\cap N$ be any element.

$\Rightarrow x \in HK$ and $x \in N$.

$\Rightarrow x = hk$ and $x \in N$ $x = n$ for some $h\in H, k\in K, n\in N$.

As $H \subseteq N$

$\Rightarrow h \in H \subseteq N$

$\Rightarrow h \in N$

i.e. $h, n \in N$ and $k = h^{-1}x$

$\Rightarrow h^{-1}n \in N$ and $h^{-1}x \in K$

$\Rightarrow h^{-1}x \in N$ and $h^{-1}x \in K$

$\Rightarrow k \in N$ and $k \in K$

$\Rightarrow k \in N\cap K$

$\Rightarrow x = hk \in H(N\cap K)$

$h\in H$, $k\in N\cap K$ $\therefore hk \in H(N\cap K)$.

$\therefore (HK) \cap N \subseteq H(K \cap N)$ —— (6)

By (5) & (6),

$(HK) \cap N = H(K \cap N)$.

This shows that $M(G)$ is modular lattice.

Q. Show that set of all normal subgroups of symmetric group on three letters (S_3), is modular lattice.

→ Let,

$S_3 = \{\rho_0, \rho_1, \rho_2, \mu_1, \mu_2, \mu_3\}$

where, $\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $\mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $\mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

$\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ $\mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

where, $H_1 = \{\rho_0\}$

$H_2 = \{\rho_0, \rho_1, \rho_2\}$

$H_3 = S_3$

are all normal subgroups of S_3.

$\therefore M(S_3) = \{H_1, H_2, H_3\}$

Clearly, $H_1 \trianglelefteq H_2 \trianglelefteq H_3$

and $\{H_1, H_2, H_3\}$ is a chain.

$\therefore$ Hasse diagram of $M(S_3)$ is,

$M(S_3)$

Here, we have to prove $M(S_3)$ is a modular lattice.
i.e. we prove,
$H_1 \trianglelefteq H_6 \Rightarrow H_1(H_3 \cap H_2) = (H_1 H_2) \cap H_3$.
Now consider,

$$H_1 (H_3 \cap H_2) = H_1 H_2$$
$$= \{\rho_0\rho_0, \rho_0\rho_1, \rho_0\rho_2\}$$
$$= \{\rho_0, \rho_1, \rho_2\}$$

$$\Rightarrow H_1 (H_3 \cap H_2) = H_2 \quad \text{—— (1)}$$

Now consider,

$$(H_1 H_2) \cap H_3 = \{\rho_0\rho_0, \rho_0\rho_1, \rho_0\rho_2\} \cap H_3$$
$$= \{\rho_0, \rho_1, \rho_2\} \cap H_3$$
$$= \{\rho_0, \rho_1, \rho_2\}$$

$$\Rightarrow (H_1 H_2) \cap H_3 = H_2 \quad \text{—— (2)}$$

$\therefore$ From (1) + (2),
$H_1(H_3 \cap H_2) = (H_1 H_2) \cap H_3$
$\therefore$ $M(S_3)$ is a modular lattice.

OR

$H_2 \trianglelefteq H_6 \Rightarrow H_2(H_3 \cap H_1) = (H_2 H_1) \cap H_3$
consider,

$$H_2 (H_3 \cap H_1) = H_2 H_1 = \{\rho_0\rho_0, \rho_1\rho_0, \rho_2\rho_0\}$$
$$= \{\rho_0, \rho_1, \rho_2\}$$

$$\therefore H_2 (H_3 \cap H_1) = H_2 \quad \text{—— (3)}$$

Now consider,

$$(H_2 H_1) \cap H_3 = (H_2 H_1) \cap H_3 = \{\rho_0\rho_0, \rho_1\rho_0, \rho_2\rho_0\} \cap H_3$$
$$= \{\rho_0, \rho_1, \rho_2\} \cap H_3$$
$$= \{\rho_0, \rho_1, \rho_2\}$$

$$\therefore (H_2 H_1) \cap H_3 = H_2 \quad \text{—— (4)}$$

$\therefore$ From (3) & (4), $H_2(H_1 \cap H_3) = (H_2 H_1) \cap H_3$
$\therefore$ $M(S_3)$ is modular lattice.

Note:-

1] Sublattice of modular (distributive) lattice is again modular (Distributive).

2] Homomorphic image of modular (distributive) lattice is again modular (distributive).

Q. Prove that sublattice of modular lattice is modular.

→ Proof:-

Let L be a modular lattice.
Let S be any sublattice of modular lattice L.

Claim:- S is modular.
on contrary assume that S is non-modular.
$\Rightarrow \exists\ x, y, z \in S$ such that
$x \geq z \Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee z$

$\Rightarrow$ $x, y, z \in S \subseteq L$
$\Rightarrow$ L is non-modular. ($\because$ above property fails)

$\Rightarrow$ which is a contradiction, to L is a modular lattice.
Hence our assumption is wrong.
$\therefore$ S is modular lattice.
$\therefore$ sublattice of modular lattice is again modular.
Hence the proof.

VVI Q. Prove that sublattice of distributive lattice is again distributive.

→ Proof:-

Let L be a distributive lattice.
Let S be any sublattice of 'L'.

Claim:- S is distributive.
On contrary assume that S is non-distributive.

$\Rightarrow \exists\, x, y, z \in S$ such that,
$x \wedge (y \vee z) \neq (x \wedge y) \vee (x \wedge z)$.

$\Rightarrow x, y, z \in S \subseteq L \Rightarrow x, y, z \in L$ also.

$\Rightarrow$ L is non-distributive. ($\because$ Property fails)

which is a contradiction.
Hence our assumption is wrong.
$\therefore$ S is a distributive lattice.
$\therefore$ sublattice of distributive lattice is again distributive.

VVI Q. ~~Homo~~ Prove that homomorphic image of modular lattice is again modular.

→ Proof:-

Let L be a modular lattice.
To prove the given result,
Define a map $\phi : L \longrightarrow M$ such that $\phi(L) = M$. ϕ is homomorphism.

Claim:- M is modular.
Let, As L is modular lattice.
$x, y, z \in L$ be any elements.

$\Rightarrow x \geq z \Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee z$

We know that homomorphism is an isotone map, so we can write,

$\phi(x) \geq \phi(z) \Rightarrow \phi(x) \wedge \phi(y \vee z) = \phi(x \wedge y) \vee \phi(z)$

$\Rightarrow \phi(x) \wedge [\phi(y) \vee \phi(z)] = [\phi(x) \wedge \phi(y)] \vee \phi(z)$

($\because \phi$ is join & meet homomorphism)

$\Rightarrow$ If $\phi(x) \geq \phi(z)$

$\Rightarrow \phi(x) \wedge [\phi(y) \vee \phi(z)] = [\phi(x) \wedge \phi(y)] \vee \phi(z)]$

This shows that,

Homomorphic image of modular lattice lattice is again modular.

$\therefore \phi(L) = M$ and M is modular.

imp Q. Prove that homomorphic image is distributive lattice is distributive.

→ Proof:-

Let L be a distributive lattice.

Define a map $\phi : L \to M$ s.t.

$\phi(L) = M$, ϕ is homomorphism.

Claim:- M is distributive.

As L is distributive, $x, y, z \in L$ be any element

$\Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

$\Rightarrow \phi(x) \wedge [\phi(y \vee z)] = [\phi(x \wedge y)] \vee [\phi(x \wedge z)]$

($\because$ Homomorphism is isotone map)

$\Rightarrow \phi(x) \wedge [\phi(y) \vee \phi(z)] = [\phi(x) \wedge \phi(y)] \vee [\phi(x) \wedge \phi(z)]$

($\because \phi$ is join & met homomorphism)

$\Rightarrow$ Homomorphic image of distributive lattice is again distributive.

$\Rightarrow \phi(L) = M$, $\because$ M is distributive.

VIMP Dedekinds Modular:-

Th^m 20:-

A lattice is modular iff it has no sublattice isomorphic to N_5.

→ Proof:-

Suppose L is modular lattice.
We know that, sublattice of modular lattice is modular, and homomorphic image of modular lattice is again modular.

⇒ No sublattice of modular lattice L is isomorphic to N_5.
Bcoz N_5 is non-modular.

Conversely,

$p \to q$
$\sim q \to \sim q$

To prove that if L has no sublattice isomorphic to N_5 then L is modular.
ie.
We prove that, if L is not modular then L has sublattice isomorphic to N_5.
We know that,

By def^n of modular lattice,
For $x, y, z \in L$,
As $x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z$
Since L is non-modular,
$\Rightarrow x \leq z \Rightarrow (x \vee y) \wedge z \neq (x \vee y) \wedge z \quad x \vee (y \wedge z)$
$x \vee (y \wedge z) < (x \vee y) \wedge z$

We established that in lattice L a subset S consisting of 5 elements for a sublattice isomorphic to N_5.

Let $S = \{u, a, b, c, v\}$ where,
$u = y \wedge z$, $a = x \vee (y \wedge z)$, $b = y$,
$c = (x \vee y) \wedge z$, $v = x \vee y$

Clearly we have,
$u < a$, $a < c$, $c < v$.
i.e. $\Rightarrow$ $u < a < c < v$.

1] $c \wedge b = [(x \vee y) \wedge z] \wedge y$
$= (x \vee y) \wedge (z \wedge y)$ $\quad$ $\because$ [Associativity]
$= (x \vee y) \wedge (y \wedge z)$ $\quad$ [$\because$ Commutativity]
$= [(x \vee y) \wedge y] \wedge z$ $\quad$ [$\because$ Associativity]
$= y \wedge z$ $\quad$ [$\because$ Absorption]
$= u$

2] $a \wedge b = [x \vee (y \wedge z)] \wedge y$
$\geqslant u$

$a \wedge b = [x \vee (y \wedge z)] \wedge y \geqslant u$

Thus we obtain,
$u \leq a \wedge b$ and $a \wedge b \leq c \wedge b = u$
$\Rightarrow$ $a \wedge b = u$

3] $a \vee b = [x \vee (y \wedge z)] \vee y$
$= y \vee [x \vee (y \wedge z)]$ $\quad$ [$\because$ Commutativity]
$= (y \vee x) \vee (y \wedge z)$ $\quad$ [$\because$ Associativity]
$= v \vee u$
$= v$ $\quad$ ($\because u < v$)

4] $c \vee b = [(x \vee y) \wedge z] \vee b$
$\leq (x \vee y) \vee b = x \vee y = v$
$\therefore c \vee b \leq v$

Also, $a \vee b = v$, $c \vee b \leq v$

$a \vee b \leq c \vee b \leq v$

$\Rightarrow v \leq c \vee b \leq v$

$\Rightarrow c \vee b = v$

$\therefore v = a \vee b = c \vee b$

By (1) to (4) with $u < a < c < v$,
S is sublattice of 'L'.

Now we have to show that all elements of set S are distinct.
i.e. $u \neq b$ ~~a/b~~, $v \neq b$, $u \neq v$, $c \neq v$, $a \neq b$, $c \neq b$.

$u = a \wedge b$

Let, $u = b$

$\Rightarrow a \wedge b = b$ and $c \wedge b = u = b$

$\Rightarrow b \leq a$ and $c \wedge b = u = b$

$\Rightarrow b \leq a$ and $c \wedge b = b$

$\Rightarrow b \leq a$ and $b \leq c$

As,

$v = a \vee b = a$

Also $v = c \vee b = c$

$\Rightarrow a = c$

which is contradiction as $\underline{a < c}$.

$\therefore$ our assumption is wrong.

$\therefore \underline{u \neq b}$.

Let, $v = b$

$\Rightarrow c \vee b = b$

$\Rightarrow c \leq b$

Also,

$u = c \wedge b$

$\Rightarrow \quad u = c$

which is contradiction as $\underline{\underline{u < a < c}}$.

$\therefore$ our assumption is wrong.

$\therefore \quad \underline{\underline{v \neq b}}$.

Let $u = v$

$\Rightarrow \quad a \wedge b = a \vee b$

$\Rightarrow \quad a = b$

Also, $c \wedge b = c \vee b$

$\Rightarrow \quad c = b$

$\therefore \quad a = c$

which is contradiction, as $\underline{\underline{a < c}}$.

$\therefore$ our assumption is wrong.

$\therefore \quad \underline{\underline{u \neq v}}$

Let, $c = v$

$\Rightarrow \quad c = c \vee b$

$\Rightarrow \quad b \leq c$

$\therefore \quad b = c \wedge b = u$

$\therefore \quad b = u$

which is contradiction as $\underline{\underline{u \neq b}}$.

$\therefore$ our assumption is wrong.

$\therefore \quad \underline{\underline{c \neq v}}$.

Let, $a = b$

$\Rightarrow \quad a \wedge b = b$

$\Rightarrow \quad u = b$

which is contradiction as $\underline{\underline{u \neq b}}$.

$\therefore$ our assumption is wrong.

$\therefore \quad \underline{\underline{a \neq b}}$.

Let $c = b$

$\Rightarrow c \wedge b = b$

$\Rightarrow u = b$

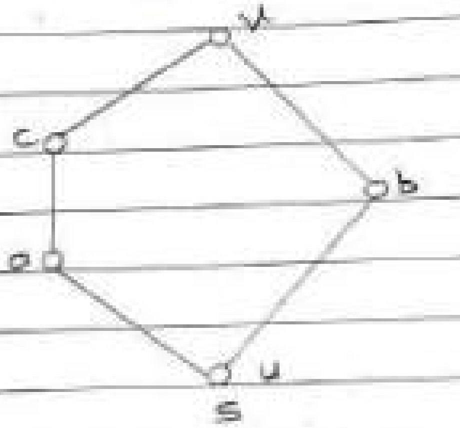
which is contradiction as $u \neq b$.

$\therefore$ our assumption is wrong.

$\therefore$ $c \neq b$.

Hence all elements of S are distinct.

$\therefore$ Hasse diagram of S is given by,



$\cong N_5$ (Pentagon)

Thus if L is non-modular lattice then there exist a sublattice isomorphic to N_5.

It's converse is also true.

i.e. if L is modular lattice then there is no sublattice isomorphic to N_5.

i.e. there is no sublattice isomorphic to N_5 then L is modular.

Hence the result.

Lemma 8 :-

Let $x, y, z \in L$ where L is any lattice then following conditions always holds.

i) $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$

ii) $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$

iii) $(x \wedge y) \vee (x \wedge z) \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$

iv) $(x \wedge y) \vee (x \wedge z) \leq x \wedge [y \vee (x \wedge z)]$.

→ Proof:-

Let L be any lattice.

Let, $x, y, z \in L$ be any elements.

i) We know that,

$x \wedge y \leq x$ and $x \wedge y \leq y \leq y \vee z$

$\Rightarrow x \wedge y \leq x$ and $x \wedge y \leq y \vee z$

$\Rightarrow (x \wedge y) \wedge (x \wedge y) \leq x \wedge (y \vee z)$

$\Rightarrow x \wedge y \leq x \wedge (y \vee z)$ —— ①

Simillarly, We have

$x \wedge z \leq x$ and $x \wedge z \leq z \leq y \vee z$

$\Rightarrow x \wedge z \leq x$ and $x \wedge z \leq y \vee z$

$\Rightarrow (x \wedge z) \wedge (x \wedge z) \leq x \wedge (y \vee z)$

$\Rightarrow (x \wedge z) \leq x \wedge (y \vee z)$ —— ②

By ① and ②

$(x \wedge y) \vee (x \wedge z) \leq [x \wedge (y \vee z)] \vee [x \wedge (y \vee z)]$

$\Rightarrow (x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$ $\forall x, y, z \in L$

ii) We know that,

$x \leq x \vee y$ and $x \leq x \vee z$

$\Rightarrow x \wedge x \leq (x \vee y) \wedge (x \vee z)$

$\Rightarrow x \leq (x \vee y) \wedge (x \vee z)$ —— ③

Simillarly we have,

$y \leq x \vee y$ and $y \wedge z \leq y \leq x \vee z$

$\Rightarrow y \wedge z \leq y \leq x \vee y$ and $y \wedge z \leq z \leq x \vee z$

$\Rightarrow \quad y \wedge z \leq x \vee y \quad$ and $\quad y \wedge z \leq x \vee z$

$\Rightarrow \quad (y \wedge z) \wedge (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$

$\Rightarrow \quad y \wedge z \leq (x \vee y) \wedge (x \vee z) \quad$ — ④

By ③ & ④,

$x \vee (y \wedge z) \leq [(x \vee y) \wedge (x \vee z)] \vee [(x \vee y) \wedge (x \vee z)]$

$\Rightarrow \quad x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$

iii) We know that

$\cancel{x \wedge y} \leq x \vee y$ and $x \wedge z \leq x \vee z$ and $y \wedge z \leq y \vee z$

$x \wedge y \leq x, \; y \leq x \vee y \quad$ — (*)

$x \wedge z \leq x, \; z \leq x \vee z \quad$ — (**)

$y \wedge z \leq y, \; z \leq y \vee z \quad$ — (***)

By (*) & (**)

$x \wedge y \leq (x \vee y) \wedge (x \vee z)$

By (**) & (***)

$x \wedge y \leq (x \vee z) \wedge (y \vee z)$

$\Rightarrow \quad (x \wedge y) \wedge (x \wedge y) \leq [(x \vee y) \wedge (x \vee z)] \wedge [$

$x \wedge y \leq (x \vee y) \wedge (x \vee z) \wedge (y \vee z) \quad$ — ⑤

Similarly,

$x \wedge z \leq (x \vee y) \wedge (x \vee z) \wedge (y \vee z) \quad$ — ⑥

$y \wedge z \leq (x \vee y) \wedge (x \vee z) \wedge (y \vee z) \quad$ — ⑦

By ⑤, ⑥, ⑦,

$(x \wedge y) \vee (x \wedge z) \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$

iv) We know that,

$x \leq x \quad$ and $\quad y \leq y \vee (x \wedge z)$

$\Rightarrow x \wedge y \leq x \wedge [y \vee (x \wedge z)]$ —— ①

We know that,
$x \wedge z \leq x$ and $x \wedge z \leq y \vee (x \wedge z)$
$\Rightarrow (x \wedge z) \wedge (x \wedge z) \leq x \wedge [y \vee (x \wedge z)]$
$\Rightarrow x \wedge z \leq x \wedge [y \vee (x \wedge z)]$ —— ②

$\therefore$ From ① & ②,
$(x \wedge y) \vee (x \wedge z) \leq x \wedge [y \vee (x \wedge z)] \vee x \wedge [y \vee (x \wedge z)]$
$\Rightarrow (x \wedge y) \vee (x \wedge z) \leq x \wedge [y \vee (x \wedge z)]$.

Birkhoff's Distributive Criteria :-
Th^m 21 :-

A lattice is distributive iff it has no sublattice isomorphic to M_3 or N_5.

→ Proof :-

Let L be any lattice.

Let L is distributive lattice.

We know that, sublattice of distributive lattice is distributive.

Let S is any sublattice of distributive lattice L.

$\Rightarrow$ S is distributive.

$\Rightarrow$ S is not isomorphic to M_3 or N_5.

Because M_3 and N_5 are not distributive.

i.e. L has no sublattice isomorphic to M_3 or N_5.

Conversely,

A lattice L does not have any sublattice isomorphic to M_3 or N_5.

Then we have to prove L is distributive lattice.

In view of Dedekinds modular we only need to prove that if L is modular but not distributive then L has sublattice isomorphic to M_3.

We know that a lattice L is distributive iff $\forall\ a, b, c \in L$ satisfies median.

Let L is not distributive lattice then $\exists\ p, q, r \in L$ such that,

$(p\wedge q)\vee(q\wedge r)\vee(r\wedge p) < (p\vee q)\wedge(q\vee r)\wedge(r\vee p)$

Let,

$u = (p\wedge q)\vee(q\wedge r)\vee(r\wedge p)$

$v = (p\vee q)\wedge(q\vee r)\wedge(r\vee p)$

$a = u\vee(p\wedge v) = (u\vee p)\wedge v$ (∵ Modular property)

$b = u\vee(q\wedge v) = (u\vee q)\wedge v$

$c = u\vee(r\wedge v) = (u\vee r)\wedge v$

Clearly, $u < v$

Let $S = \{u, a, b, c, v\}$

Let,

$a\wedge b = [(u\vee p)\wedge v]\wedge[(u\vee q)\wedge v]$

$= [(u\vee p)\wedge v]\wedge[v\wedge(u\vee q)]$

$= [(u\vee p)\wedge(v\wedge v)\wedge(u\vee q)]$

$= (u\vee p)\wedge(u\vee q)\wedge v$

$= \{(p\wedge q)\vee(q\wedge r)\vee(r\wedge p)\vee p\}\wedge(u\vee q)\wedge v$

$= \{(p\wedge q)\vee(q\wedge r)\vee p\}\wedge(u\vee q)\wedge v$

$= \{p\vee(q\wedge r)\}\wedge(u\vee q)\wedge v$

$= \{p\vee(q\wedge r)\}\wedge\{(p\wedge q)\vee(q\wedge r)\vee(r\wedge p)\vee q\}\wedge v$

$= \{p\vee(q\wedge r)\}\wedge\{(p\wedge q)\vee q\vee(r\wedge p)\}\wedge v$

$= \{p\vee(q\wedge r)\}\wedge\{q\vee(r\wedge p)\}\wedge v$

$= \{[((q\wedge r)\vee p)\wedge q]\vee(r\wedge p)\}\wedge v$

$= \{(q\wedge r) \vee (p\wedge q) \vee (r\wedge p)\} \wedge v$

$= u \wedge v$

$= u \qquad (\because u < v)$

similarly,

$b\wedge c = [(u\vee q)\wedge v] \wedge [(u\vee r)\wedge v]$

$= [(u\vee q)\wedge v] \wedge [v \wedge (u\vee r)]$

$= (u\vee q) \wedge (v\vee v) \wedge (u\vee r)$

$= (u\vee q) \wedge (u\vee r) \wedge v$

$= \{(p\wedge q) \vee (q\wedge r) \vee (r\wedge p) \vee q\} \wedge$
$\{(p\wedge q) \vee (q\wedge r) \vee (r\wedge q) \vee r\} \wedge v$

$\cancel{= \{(p\wedge q) \vee (q\wedge r)\} \vee r\} \wedge v}$

$= \{(p\wedge q) \vee q \vee (r\wedge p)\} \wedge$
$\{(p\wedge q) \vee (q\wedge r) \vee r\} \wedge v$

$= \{q \vee (r\wedge p)\} \wedge \{(p\wedge q) \vee r\} \wedge v$

$= \{(r\wedge p) \vee q\} \wedge \{r \vee (p\wedge q)\} \wedge v$

$= \{(r\wedge p) \vee (q\wedge r) \vee (p\wedge q)\} \wedge v$

$= \{(p\wedge q) \vee (r\wedge p) \vee (q\wedge r)\} \wedge v$

$= u \wedge v$

$b\wedge c = u \qquad (\because u < v)$

Similarly, $a\wedge c = u$

$\therefore \quad u = a\wedge b = b\wedge c = a\wedge c \qquad$ — ①

Let,

$\cancel{a\vee b = [(u\vee p)\wedge v] \vee [(u\vee q)\wedge v]}$

$a\vee b = [u\vee (p\wedge v)] \vee [u\vee (q\wedge v)] = u\vee (p\wedge v)\vee(q\wedge v)$

$= u\vee \{p\wedge (q\vee r)\} \vee \{(r\vee p)\wedge q\}$

$= u\vee \{(q\vee r)\wedge p\} \vee \{q\wedge (r\vee p)\}$

$= u\vee \{(q\vee r) \wedge (p\vee q) \wedge (r\vee p)\}$

$= u\vee v$

$a\vee b = v$

$$
\begin{aligned}
b \vee c &= [u \vee (q \wedge r)] \vee [u \vee (r \wedge r)] \\
&= u \vee (q \wedge r) \vee (r \wedge r) \\
&= u \vee \{q \wedge (p \vee q) \wedge (q \vee r) \wedge (r \vee p)\} \vee \\
&\qquad \{r \wedge (p \vee q) \wedge (q \vee r) \wedge (r \vee p)\} \\
&= u \vee \{q \wedge (q \vee r) \wedge (r \vee p)\} \vee \{r \wedge (p \vee q) \wedge (r \vee p)\} \\
&= u \vee \{q \wedge (r \vee p)\} \vee \{r \wedge (p \vee q)\} \\
&= u \vee \{(r \vee p) \wedge q\} \vee \{r \wedge (p \vee q)\} \\
&= u \vee \{(r \vee p) \wedge (q \vee r) \wedge (p \vee q)\} \\
&= u \vee v \\
&= v
\end{aligned}
$$

$\therefore$ $b \vee c = v$ and $a \vee c = v$

$\therefore$ $v = a \vee b = b \vee c = a \vee c$ ——②

$\therefore$ By ① + ②, S is sublattice of L.

By observation we have

$u \leq a \leq v$, $a \leq b \leq v$, $u \leq c \leq v$.

We show that the elements of S are distinct.

① Let $u = a$

$\Rightarrow$ $a \wedge b = a$

$\Rightarrow$ $a \leq b$

Also $a \wedge c = a$

$\Rightarrow$ $a \leq c$

Let. $v = c \vee a$

$v = c$

$v = a \vee b = b$

$\Rightarrow$ $b = c$

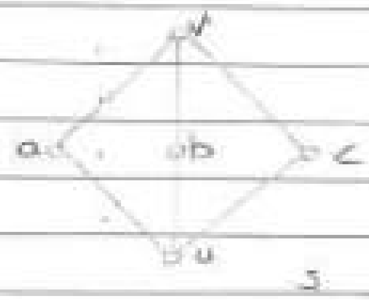
But, $u = b \wedge c = b = b \vee c = v$

ie. $u = v$

which is contradiction as $(u<v)$
$\Rightarrow \quad u \neq a$

② similarly we show $u \neq b$, $u \neq c$
By dual argument we show $v \neq a$, $v \neq b$, $v \neq c$.

③ Let $a = b$
$a \wedge b = a$
$u = a$
which is a contradicts to ① $u \neq a$
$\Rightarrow \quad a \neq b$
$a \neq c$ & $b \neq c$
By ①, ②, ③ all elements are distinct.
Hasse diagram of S is



$\cong M_3$

i.e. If L is non-distributive lattice $\exists$ a sublattice isomorphic to M_3.
Therefore converse of above statement is also true. i.e. If S is sublattice of L not isomorphic to M_3 then L is distributive lattice.

Relatively Complemented Lattice:-
A lattice is said to be relatively complemented lattice if every element in any interval containing it have relative complement

in it ie. if $[a, b] \subseteq L$.

If $c \in [a, b]$ then $\exists\, d \in [a, b]$ such that $c \wedge d = a$ and $c \vee d = b$.

Th^m 22:-

In a bounded distributive lattice if an element a has complement then it has relative complement in any interval containing it.

→ Proof:-

Let L is bounded distributive lattice.

Let $a \in L$ has complement.

Let 'x' be the complement of 'a'.

$\Rightarrow a \wedge x = 0$

$a \vee x = 1$

Let $a \in [b \ c] \subseteq L$

Claim:- a has relative complement in $[b \ c]$

We construct

$y = (b \vee x) \wedge c \in [b \ c]$

consider,

$$\begin{aligned}
a \wedge y &= a \wedge [(b \vee x) \wedge c] \\
&= a \wedge [c \wedge (b \vee x)] && (\because \text{Commutative}) \\
&= (a \wedge c) \wedge (b \vee x) && (\because \text{Associative}) \\
&= a \wedge (b \vee x) && (\because b \leq a \leq c) \\
&= (a \wedge b) \vee (a \wedge x) && (\because \text{Distributive}) \\
&= b \vee (a \wedge x) \\
&= b \vee 0
\end{aligned}$$

$\therefore a \wedge y = b$

Similarly,

$$
\begin{aligned}
a \vee y &= a \vee [(b \vee x) \wedge c] \\
&= a \vee [c \wedge (b \vee x)] && (\because \text{Commutative}) \\
&= (a \vee c) \wedge [a \vee (b \vee x)] && (\because \text{Distributive}) \\
&= c \wedge [(a \vee b) \vee x] && (\because b \le a \le c) \\
&= c \wedge (a \vee x) && (\because b \le a \le c) \\
&= c \wedge 1 && (a \vee x = 1) \\
\therefore a \vee y &= c
\end{aligned}
$$

$\therefore$ $a \wedge y = b$, $a \vee y = c$

This shows that, y is relative complement of 'a'.

Hence the result.

Boolean Lattice:-

A lattice L is called Boolean lattice if it is complemented and distributive.

Complete lattice:-

A lattice L is called complete lattice if $\wedge H$ and $\vee H$ exists for every $\phi \neq H \subseteq L$.

Note:-

1] Every bounded lattice is complete lattice.

2] Every boolean lattice is complete lattice.

→ Proof:-

Let 'L' be a boolean lattice

⇒ 'L' is complemented and distributive.

⇒ L' must be bounded.

⇒ 'L' is complete lattice.

But converse may or may not be true.

3) Every Boolean lattice is relatively complemented lattice.

→ <u>Proof</u>:-

complement may or may not be [illegible].

We know that in a bounded distributive lattice an element has unique complement. Also every element has relative complement in any interval containing it.
Hence Boolean lattice is relatively complemented lattice.

Lemma 9:-

If P be a poset, $\wedge H$ exists for all $\phi \neq H \subseteq P$ then P is complete lattice.

→ <u>Proof</u>:-

Let $\phi \neq H \subseteq P$ be any subset.

Let $K = H^u = \{ x \in P / a \leq x \ \forall a \in H \}$

$= \text{set of all upper bounds of } H$

Clearly, $\phi \neq K \subseteq P$

Then by hypothesis $\wedge K$ exists

$\Rightarrow$ lub of H exists.

$\Rightarrow \wedge K = a = \vee H$

Thus for any $\phi \neq H \subseteq P$,

If $\wedge H$ exists then $\vee H$ exists.

$\Rightarrow P$ is complete lattice.

Note:-

1) $I(L)$ and $con(L)$ are complete lattices.

Birkhoff's Distributive Criteria:-

Corollary:-

A bounded lattice 'L' is distributive iff every element has atmost one relative complement in any interval containing it.

→ Proof:-

We know that in bounded distributive lattice an element has unique complement.

Also we know that in bounded distributive lattice if an element a has complement then it has relative complement in any interval containing it ie.

ie. if $a \in [c, d] \subseteq L$ and y is relative complement of 'a' then we have,

$y = (b \vee x) \wedge c \in [c, d]$ is unique.

where 'x' is complement of 'a'.

Therefore, every element has atmost one relative complement in any interval containing it.

Conversely,

Suppose that every element in lattice L has atmost one relative complement in any interval containing it.

Claim:- 'L' is distributive lattice.

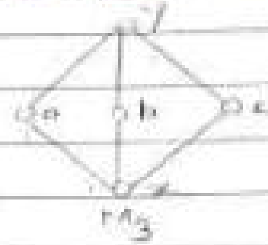
Let L is non-distributive lattice.

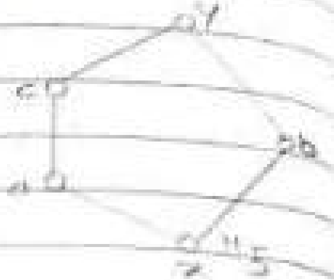
$\Rightarrow$ By Birkhoff's Distributive criterian,

$\exists$ a sublattice of L which is isomorphic to M_3 or N_5.

$\Rightarrow \exists$ sublattice S of L which is isomorphic to M_3 or N_5.

⇒ S is isomorphic to M_3 or N_5 re.

$S \cong$ 

or $S \cong$ 

⇒ Each of this diagram an element 'b' me have relative complement.
Relative complements are $a, c \in [x, y]$ which is contradiction to our hypothesis.
Thus our assumption is wrong.
∴ 'L' is distributive lattice.

Lemma 10 :-

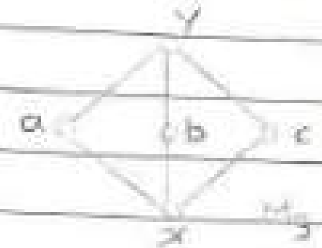
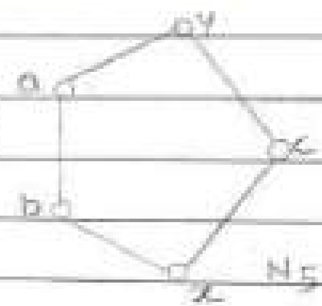
A lattice 'L' is distributive iff for any two ideals I and J of 'L', $I \vee J = \{ i \vee j \mid i \in I, j \in J \}$

→ Proof:-

Let L is distributive lattice.
Let I and J be any two ideals of L.
We have,
$I \vee J = \mathcal{I}(I \cup J]$
Let $t \in I \vee J$ be any element
⇒ $t \leq i \vee j$ for some $i \in I$, $j \in J$
⇒ $t = t \wedge (i \vee j)$
⇒ $t = \underset{a}{(t \wedge i)} \vee \underset{b}{(t \wedge j)}$ (∵ 'L' is Distributive)
⇒ $t \wedge i \in I$ and $t \wedge j \in J$
⇒ $t = a \vee b$ such that $a \in I$, $b \in J$
∴ $I \vee J = \{ i \vee j \mid i \in I, j \in J \}$.

Conversely,
Let $I \vee J = \{a \vee b \mid a \in I,\ b \in J\}$

<u>Claim:-</u> 'L' is distributive lattice.
On contrary assume that 'L' is non-distributive lattice.
Then by Birkhoff's Distributive criteria, 'L' has sublattice which is isomorphic to M_3 or N_5.
i.e. S is sublattice of L such that

$S \cong$  or $S \cong$ 

Let, $I = (b]$ and $J = (c]$.
Let $a \in I \vee J$
$\Rightarrow$ $a = i_1 \vee j_1$ for some $i_1 \in I$ & $j_1 \in J$
$\Rightarrow$ $i_1 \leq a$ and $j_1 \leq a$ also $j_1 \leq c$.
$\Rightarrow$ $j_1 \leq a \wedge c < b$
$\Rightarrow$ $j_1 \in (b]$
$\Rightarrow$ $j_1 \in I$
$\Rightarrow$ $a = i_1 \vee j_1 \in I$
$\Rightarrow$ $a \in I$
$\Rightarrow$ $a \leq b$
which is impossible, hence our assumption is wrong.
$\therefore$ 'L' is distributive lattice.

Lemma 11:-
Let 'I' and J be ideals of distributive lattice 'L'. If $I \wedge J$ and $I \vee J$ are principle ideals then so I and J.

→ Proof:-
Let 'L' be a distributive lattice.
Let 'I' and 'J' be any ideals of lattice. Let $I \wedge J$ and $I \vee J$ are principle ideals of L.

Claim:- I and J are principle ideals.
As $I \wedge J$ is principle ideal.
$\Rightarrow$ $I \wedge J = (x]$ for some $x \in L$.
Similarly,
As $I \vee J$ is principle ideal.
$\Rightarrow$ $I \vee J = (y]$ for some $y \in L$.
Thus,
$x = i \wedge j$ and $y = i \vee j$ for some $i \in I, j \in J$

Let $c = x \vee i$ and $b = x \vee j$

Subclaim:- $I = (c]$ and $J = (b]$
As $x \in I \wedge J = I \cap J$
$\Rightarrow$ $x \in I$ and $x \in J$
$\Rightarrow$ $x \vee i \in I$ ($\because$ I is ideal)
$\Rightarrow$ $c \in I$
Similarly,
$b \in J$
Assume that $J \neq (b]$
$\Rightarrow \exists\, a \in J$ such that $b < a$
consider the set $S = \{x, a, b, c, y\}$

Here, $x \leq x \vee i = c \leq y$

As $i \leq i \vee j$

$\Rightarrow \quad x \vee i \leq x \vee (i \vee j)$

$\Rightarrow \quad c \leq x \vee (i \vee j)$

$\Rightarrow \quad c \leq (i \wedge j) \vee (i \vee j)$

$\Rightarrow \quad c \leq i \vee j$

$\Rightarrow \quad c \leq y$

Simillarly,

$x \leq x \vee j = b < a \leq y$

i) $b \wedge c = (x \vee j) \wedge (x \vee i)$

$= x \vee (i \wedge j)$ (∵ L is distributive

$= x \vee x$ (∵ idempotent

$= x$ (∵ Idempotent)

ii) $a \wedge c = a \wedge (x \vee i)$

$= (a \wedge x) \vee (a \wedge i)$ (∵ L is distributive

$= x \vee i$

$= x$

iii) $b \vee c = (x \vee j) \vee (x \vee i)$

$= (x \vee j) \vee (i \vee x)$ (∵ commutative)

$= x \vee (j \vee i) \vee x$

$= x \vee (j \vee i)$ $(x \vee x = x$

$= (i \wedge j) \vee (j \vee i)$ $(x = i \wedge j)$

$= j \vee i$

$= y$

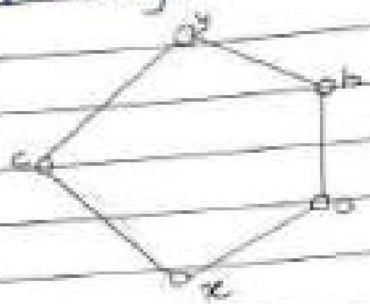
iv) $a \vee c = a \vee (x \vee i)$

$= y$

∴ S is sublattice of L.

join & meet of all elements of S again in S

∴ Hasse diagram of S is,



$\cong N_5$

Birkhoff's Th. for Th.

which is contradiction to L is distributive lattice.

Hence our assumption is wrong.

Therefore $J = (b]$.

This shows that, J is principle ideal.

Similarly, $I = (c]$,

Hence I is principle ideal.

Th^m 23 :-

Let L is distributive lattice, $a \in L$ be any element then the map $\phi : L \to (a] \times [a)$ is an embedding map where $\phi(x) = (x \wedge a, x \vee a)$ ∀ for all $x \in L$ also ϕ is isomorphism if 'a' has complement in L.

→ <u>Proof</u>:-

'L' is distributive lattice.

Let $a \in L$ be any element

Define $\phi : L \longrightarrow (a] \times [a)$ by

$\phi(x) = (a \wedge x, \; a \vee x) \qquad \forall \; x \in L.$

1) To prove ϕ is one-one.

Let, $\phi(x) = \phi(y)$ $\qquad x, y \in L$

$\Rightarrow (x \wedge a, \; x \vee a) = (y \wedge a, \; y \vee a)$ $\qquad$ (∵ By def^n)

$\Rightarrow x \wedge a = y \wedge a$ and $x \vee a = y \vee a$.

$\Rightarrow x = y$

$\Rightarrow$ ϕ is one-one map.

2) To prove ϕ is homomorphism.

Let, $x, y \in L$ be any elements

i) $\phi(x \wedge y)$

$= [(x \wedge y) \wedge a, (x \wedge y) \vee a]$

$= [(x \wedge y) \wedge (a \wedge a), (x \vee a) \wedge (y \vee a)]$

($\because$ Idempotent & L is distrib.)

$= [(x \wedge a) \wedge (y \wedge a), (x \vee a) \wedge (y \vee a)]$

($\because$ Associative & Commutative)

$= [(x \wedge a), (x \vee a)] \wedge [(y \wedge a), (y \vee a)]$

$= \phi(x) \wedge \phi(y)$

$\therefore \phi(x \wedge y) = \phi(x) \wedge \phi(y) \quad \forall x, y \in L$

$\therefore$ ϕ is meet homomorphism.

ii) $\phi(x \vee y)$

$= [(x \vee y) \wedge a, (x \vee y) \vee a]$

$= [(x \wedge a) \vee (y \wedge a), (x \vee y) \vee (a \vee a)]$

$= [(x \wedge a) \vee (y \wedge a), (x \vee a) \vee (y \vee a)]$

$= [(x \wedge a) \vee (x \vee a)] \vee [(y \wedge a) \quad (y \vee a)]$

$= \phi(x) \vee \phi(y)$

$\therefore$ ϕ is join homomorphism.

$\therefore$ By (i) & (ii)

ϕ is homomorphism.

$\therefore$ By ① & ②,

ϕ is embedding map.

Let b is complement of 'a'.

Claim:- To prove ϕ is onto.

Let $(x, y) \in (a] \times [a)$ be any element

$\Rightarrow x \leq a$ and $a \leq y$

Consider,

$u = (x \vee b) \wedge y \in L$

$\Rightarrow \phi(u) = \phi[(x \vee b) \wedge y]$

$= \{[(x \vee b) \wedge y] \wedge a \quad [(x \vee b) \wedge y] \vee a\}$

$= \{(x \vee b) \wedge (y \wedge a) \quad a \vee [(x \vee b) \wedge y]\}$

b complement of a
$a \wedge b = 0$
$a \vee b = 1$

$= \{(x \vee b) \wedge a \quad [a \vee (x \vee b)] \wedge [(a \vee y)]\}$

$= \{(x \wedge a) \vee (b \wedge a) \quad (a \vee x) \vee b \wedge y\}$

$= \{(x \wedge a) \vee 0 \quad (a \vee b) \wedge y\}$

$= (x \wedge a, \ 1 \wedge y)$

$\phi(u) = (x, y)$

This shows that preimage of (x, y) is u.

$\therefore$ ϕ is onto.

Hence if element a has complement then ϕ is isomorphism.

Q) Show that in a distributive lattice 'L' if 'b' is complement of 'a' then b is the largest element of L such that $b \wedge a = 0$.

→ **Proof:-**

Let L is distributive lattice.
Let b is complement of 'a'.
Let if possible $\exists\ x \in L$ such that $x \wedge a = 0$

Claim:- $x \le b$

Consider, $x = x \wedge 1$

$$
\begin{aligned}
&= x \wedge (a \vee b) && (\because b \text{ is complement of } a) \\
&= (x \wedge a) \vee (x \wedge b) && (\because L \text{ is distri}) \\
&= 0 \vee (x \wedge b) \\
x &= x \wedge b \\
\Rightarrow\ x &\le b
\end{aligned}
$$

This shows that b is the largest element of L such that $b \wedge a = 0$.

Pseudo Complement:-

An element a^* is Pseudo complement of $a \in L$ if $a \wedge a^* = 0$ and $a \wedge x = 0 \Rightarrow x \le a^*$.

e.g.

Consider the lattice

Here, $a \wedge a^* = 0$ and $a \wedge x = 0$ with $x \le a^*$
$\Rightarrow a^*$ is pseudo complement of a.

Thm 24:-

If an element has pseudo complement then show that it is unique. or

OR

In a lattice show that pseudo complement of an element is unique.

→ Proof:-

Let L be any lattice.

Let $a \in L$ be any element with a has pseudo complement.

Let if possible a has two pseudo complements, b^*, c^*.

$\Rightarrow a \wedge b^* = 0$ —① and $a \wedge c^* = 0$ —②

Claim:- $b^* = c^*$

~~Let~~ As b^* is pseudo complement of a then by ① + ②,

$c^* \leq b^*$ —③

As c^* is pseudo complement of a then by ① + ②,

$b^* \leq c^*$ —④

By ③ + ④,

$b^* = c^*$.

This shows that a has unique pseudo complement.

Hence the result.

Pseudo complemented lattice:-

A meet semilattice with zero is called pseudo complemented lattice if every element has pseudo complement.

Note:-

1) Pseudo complemented lattice is denoted by S(L).

Atom:-

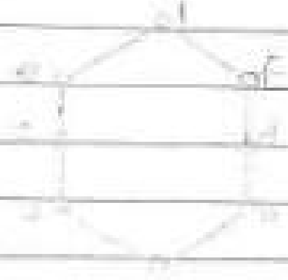
An element $a \in L$ where L is lattice with zero is called an atom if $0 \prec a$ or 'a' covers zero(0).

Co-atom:- (Dual atom)

An element $a \in L$ where L is lattice with one ('1') is called co-atom if $a \prec 1$ or '1' covers 'a'.

eg.

Consider the following lattice,



Clearly,

Here, a, b are atoms.

e, f are co-atoms.

Atomic lattice:-

A lattice L with zero element is called atomic lattice if for any $x \in L$ $\exists$ an atom $a \in L$ such that $a \leq x$.

Co-atomic lattice:-

A lattice L with one is co-atomic lattice if for any $x \in L$ $\exists$ an co-atom $a \in L$ such that $x \leq a$.

Note:-

1) Every finite lattice is atomic lattice.

2) Every finite lattice is co-atomic lattice.

Pseudo complemented lattice:-

A lattice is called pseudo complemented lattice if every element has pseudo complem

Th^m 25:-

Let 'L' be a Pseudo Complemented meet semilattice. Consider $S(L) = \{a^* \mid a \in L\}$. The partial ordering of L mayke S(L) into lattice and S(L) is Boolean lattice. For any $a, b \in S(L)$ we define $a \wedge b = \inf\{a, b\} \in S(L)$ and

$$a \vee b = (a^* \wedge b^*)^*.$$

or

With usual notations prove that S(L) is Boolean lattice.

→ Proof:-

We will prove the following.

i) $a \leq a^{**}$ $\quad \forall a \in L$

ii) If $a \leq b$ then $b^* \leq a^*$

iii) $a^* = a^{***}$

iv) $a \in S(L)$ iff $a = a^{**}$

v) If $a, b \in S(L)$ then $a \wedge b \in S(L)$

vi] If $a, b \in S(L)$ then $a \vee b \notin S(L)$, $= (a^* \wedge b^*)^*$

i] We know that,

$a \wedge a^* = 0$

Simillarly,

$a^* \wedge (a^*)^* = 0$

But we ~~kno~~ def^n of pseudo complement.

$a \leq a^{**}$

ii] Let $a \leq b$

$a \wedge a^* = 0$

$\therefore\ a \wedge b^* = 0$

$\therefore\ b^* \leq a^*$

Claim:- $a \wedge b^* = 0$

But $a \wedge b^* = (a \wedge b) \wedge b^*$ $\quad$ ($\because a \leq b \Rightarrow a \wedge b = a$)

$= a \wedge (b \wedge b^*)$ $\quad$ ($\because$ Associative)

$= a \wedge 0$

$\therefore\ a \wedge b^* = 0$ and $a \wedge a^* = 0$

a^* is pseudo complement of a.

$\Rightarrow\ b^* \leq a^*$.

$\therefore$ If $a \leq b$ then $b^* \leq a^*$.

iii] Claim:- $a^* = a^{***}$

We know that,

$a \leq a^{**}$

(And $a \leq b \Rightarrow b^* \leq a^*$)

$\therefore\ a^{***} \leq a^*$ $\quad$ —①

Clearly,

$a^* \leq (a^*)^{**} \Rightarrow a^* \leq a^{***}$ $\quad$ —②

$\therefore$ By ① & ②,

$a^* = a^{***}$

iv] Let $a \in S(L)$ be any element.

$\Rightarrow a = b^*$ for some $b \in L$.

$\Rightarrow a^{**} = (b^{*})^{**}$

$\Rightarrow a^{**} = b^{***}$

$\Rightarrow a^{**} = b^{*}$ (∵ By (iii))

$\Rightarrow a^{**} = a$

Conversely,

Let $a^{**} = a$ then

$(a^{*})^{*} = a$

~~then $b = a^{*}$~~ * Let consider $a^{*} = b \in L$

$\Rightarrow \quad a^{**} = b^{*} \in S(L)$.

$\Rightarrow \quad a \in S(L)$.

$\therefore a \in S(L)$ iff $a = a^{**}$.

v) Let $a, b \in S(L)$ be any elements.

$\Leftrightarrow a = a^{**}$ and $b = b^{**}$

Claim:- $a \wedge b \in S(L)$, or $(a \wedge b)^{**} = a \wedge b$

We know that,

$a \leq a^{**}$

$\Rightarrow (a \wedge b) \leq (a \wedge b)^{**}$ —③

As $a \wedge b \leq a$

$\Rightarrow a^{*} \leq (a \wedge b)^{*}$ (By (ii))

$\Rightarrow (a \wedge b)^{**} \leq a^{**}$ (By (ii)).

$\Rightarrow (a \wedge b)^{**} \leq a$ (∵ $a \in S(L)$)

Similarly,

As $(a \wedge b) \leq b$

$\Rightarrow b^{*} \leq (a \wedge b)^{*}$ (By (ii))

$\Rightarrow (a \wedge b)^{**} \leq b^{**}$ (By (ii))

$\Rightarrow (a \wedge b)^{**} \leq b$ (∵ $b \in S(L)$)

$\therefore (a \wedge b)^{**} \wedge (a \wedge b)^{**} \leq a \wedge b$

$\therefore (a \wedge b)^{**} \leq a \wedge b$ —④

By ③ & ④, $(a \wedge b)^{**} = a \wedge b$.

$\Rightarrow a \wedge b \in S(L)$ $\forall a, b \in S(L)$

avb

vi] <u>Claim</u> :- $\sup_{S(L)}\{a,b\} = (a^* \wedge b^*)^*$

As $a, b \in S(L)$ be any elements

$\Rightarrow \quad a = a^{**}$ and $b = b^{**}$

As $a^* \wedge b^* \leq a^*$

$\Rightarrow \quad (a^*)^* \leq (a^* \wedge b^*)^*$ (By (ii))

$\Rightarrow \quad a^{**} \leq (a^* \wedge b^*)^*$

$\Rightarrow \quad a \leq (a^* \wedge b^*)^*$ ($\because a \in S(L)$)

Similarly,

As $a^* \wedge b^* \leq b^*$

$\Rightarrow \quad b^{**} \leq (a^* \wedge b^*)^*$ (By (ii))

$\Rightarrow \quad b \leq (a^* \wedge b^*)^*$ ($\because b \in S(L)$)

$\therefore \ (a^* \wedge b^*)^*$ is an upper bound of a and b.

<u>Subclaim</u> :- $(a^* \wedge b^*)^*$ is lub$\{a, b\}$.

Let,

$x \in S(L)$ be any other upper bound of a and b.

$\Rightarrow \quad a \leq x$ and $b \leq x$

$\Rightarrow \quad x^* \leq a^*$ and $x^* \leq b^*$ (By (ii))

$\Rightarrow \quad x^* \wedge x^* \leq a^* \wedge b^*$

$\Rightarrow \quad x^* \leq a^* \wedge b^*$

$\Rightarrow \quad (a^* \wedge b^*)^* \leq x^{**}$ (By (ii))

$\Rightarrow \quad (a^* \wedge b^*)^* \leq x$ ($\because x \in S(L)$)

$\therefore \ (a^* \wedge b^*)^*$ is lub of a and b.

$\therefore \quad a \vee b = \sup_{S(L)}\{a, b\} = (a^* \wedge b^*)^*$.

Hence $\forall \, a, b \in S(L)$ we get,

$a \wedge b \in S(L)$ and $a \vee b = S(L)$.

$\Rightarrow \quad S(L)$ is lattice.

To prove $S(L)$ is Boolean Lattice.

① **Claim:-** Every element $a \in S(L)$ has a complement a^*.

Clearly, we have.

$a \wedge a^* = 0$ and $a \vee a^* = (a^* \wedge a^{**})^*$ $\quad (\because (a \vee b) = (a^* \wedge b^*)^*)$

$$a \vee a^* = (a^* \wedge a^{**})^* \quad (\because a \in S(L))$$
$$= (a^* \wedge a)^*$$
$$= 0^*$$
$$a \vee a^* = 1$$

$\therefore$ $a \wedge a^* = 0$ and $a \vee a^* = 1$

$\therefore$ Complement of a is a^*. $\forall a \in S(L)$.

$\Rightarrow$ $S(L)$ is complemented lattice.

② **Claim:-** $S(L)$ is distributive lattice.

ie, $(x \vee y) \wedge z \leq x \vee (y \wedge z)$. $\forall x, y, z \in L$

Subclaim:- For any $a, b \in S(L)$,

$a \leq b$ iff $a \wedge b^* = 0$

Suppose,

$a \leq b$

$\Rightarrow$ $a \wedge b^* = (a \wedge b) \wedge b^*$

$\Rightarrow$ $a \wedge b^* = a \wedge (b \wedge b^*)$

$\Rightarrow$ $a \wedge b^* = a \wedge 0$

$\Rightarrow$ $a \wedge b^* = 0$

Conversely,

Suppose, $a \wedge b^* = 0$

$\Rightarrow$ $a \leq b^{**}$

$\Rightarrow$ $a \leq b$

$\therefore$ $a \leq b$ iff $a \wedge b^* = 0$.

Consider, $x, y, z \in S(L)$ be any elements.

As $x \wedge z \leq x \vee (y \wedge z)$

$\Rightarrow (x \wedge z) \wedge [x \vee (y \wedge z)]^* = 0$ $\left(\because a \leq b \atop a \wedge b^* = 0\right)$

As $y \wedge z \leq x \vee (y \wedge z)$

$\Rightarrow (y \wedge z) \wedge [x \vee (y \wedge z)]^* = 0$ $\left(\because a \leq b \atop a \wedge b^* = 0\right)$

But $(x \wedge z) \wedge [x \vee (y \wedge z)]^*$

$= x \wedge [z \wedge (x \vee (y \wedge z))^*] = 0$ ($\because$ Associative)

— ⑤

$\Rightarrow z \wedge (x \vee (y \wedge z))^* \leq x^*$ ($\because$ def. of pseudocomplement)

But lly, $(y \wedge z) \wedge [x \vee (y \wedge z)]^* = y \wedge [z \wedge (x \vee (y \wedge z))^*] = 0$ ($\because$ Associative)

$\Rightarrow z \wedge (x \vee (y \wedge z))^* \leq y^*$ — ⑥

By ⑤ & ⑥,

$[z \wedge (x \vee (y \wedge z))^*] \wedge [z \wedge (x \vee (y \wedge z))^*] \leq x^* \wedge y^*$

$\Rightarrow z \wedge (x \vee (y \wedge z))^* \leq x^* \wedge y^*$

$\Rightarrow z \wedge (x \vee (y \wedge z))^* \wedge (x^* \wedge y^*)^* = 0.$

$\Rightarrow \{z \wedge (x^* \wedge y^*)^*\} \wedge \{x \vee (y \wedge z)\}^* = 0$ ($\because$ commutative & Associative).

$\Rightarrow \{z \wedge (x \vee y)\} \wedge \{x \vee (y \wedge z)\}^* = 0$

$\Rightarrow z \wedge (x \vee y) \leq x \vee (y \wedge z)$ $\forall x, y, z \in S(L)$

$\Rightarrow (x \vee y) \wedge z \leq x \vee (y \wedge z)$

$\therefore$ $S(L)$ is distributive lattice.

$\therefore$ By claim ① & ②,

$S(L)$ is Boolean lattice.

Join Irreducible element:-

An element $x \in L$ is join irreducible if x is not join two different elements ie $x = b \vee c$ then either $b = x$ or $c = x$.

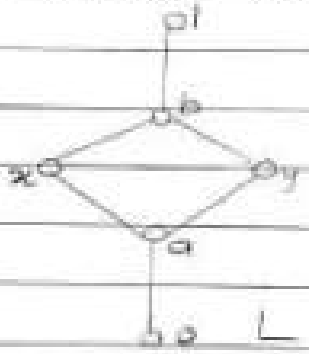
OR $\nexists$ $b, c \in L$ such that $b \neq c$ and $x = b \vee c$.

Meet Irreducible element:-

An element $y \in L$ is called meet irreducible if y is not meet of two different element ie $y = b \wedge c$ then either $b = y$ or $c = y$.

OR $\nexists$ $b, c \in L$ such that $b \neq c$ and $y = b \wedge c$.

1) Consider the lattice defined by,



L

Find join and meet irreducible elements.

→ Let,

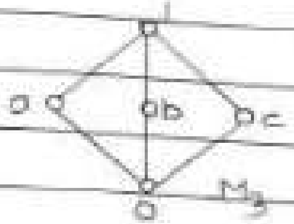
Clearly, $0, a, x, y, 1$ are join irreducible elements.

Similarly, b is join reducible element.

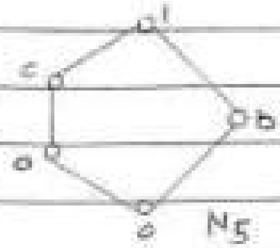
Clearly, $0, b, x, y, 1$ are meet irreducible elements.

Similarly, a is meet reducible element.

2) Find join and meet irreducible elements of M_3 and N_5.



M_3



N_5

→ Let,

For M_3 :- 0, a, b, c are join irreducible elements.
1 is join reducible element
a, b, c, 1 are meet irreducible elements
0 is meet reducible element.

For N_5 :- 0, a, b, c are join irreducible elements.
1 is join reducible element.
a, b, c, 1 are meet irreducible elements
0 is meet reducible element.

Doubly irreducible element:-

An element $x \in L$ is called doubly irreducible element if,

i) x is join irreducible.
ii) x is meet irreducible.

Note:-

1) Every atom and zero always join irreducible elements.

2) Every co-atom and one are always meet irreducible elements.

Th^m 26:-

In a Boolean lattice $0 \neq x$ is join irreducible iff x is an atom.

→ **Proof**:-

Let L be a Boolean lattice.

Let $0 \neq x \in L$ is join irreducible element.

Claim:- x is an atom.

Suppose x is not an atom.

$\Rightarrow \exists\ a \in L$ such that $0 \leq a \prec x < 1$.

As L is Boolean lattice.

$\Rightarrow \exists !$ complement of x say $x' \in L$.

there exist unique ($\exists !$)

$\Rightarrow \quad x \wedge x' = 0$ and $x \vee x' = 1$

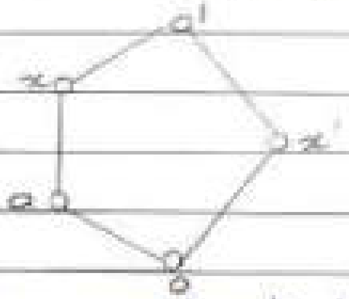
Also we must have,

$a \wedge x' = 0$ and $a \vee x' = 1$.

Let

$S = \{0, a, x, x', 1\} \subseteq L$.

which is isomorphic to N_5.

i.e. 

$\cong N_5$

Boolean ⇓ distributive ⇓ doesn't exist sublattice which is ≅ to M_3 or N_5

which is contradiction to 'L' is Boolean.

∴ Our assumption is wrong.

∴ x is atom.

Conversely,

Let x is atom.

As every atom is join irreducible.

$\Rightarrow \quad x$ is join irreducible.

Th^m 27:-

Show that in a finite lattice every element is join of join irreducible element.

→ **Proof:-**

Let 'L' is finite lattice.

Let $x \in L$ be any element.

If x is join irreducible they we are done. otherwise choose $x_1, x_2 \in L$ such that,

$x = x_1 \vee x_2$.

If x_1 and x_2 are join irreducible then we are done otherwise repeat the procedure for x_1 and x_2, continue this procedure until we will get join of irreducible element. This process must be terminate as L is finite lattice.

⇒ x can be written as join of join irreducible elements.

Note:-

Here, $1 = x \vee y$

$= (a \vee b) \vee (a \vee c)$

$= a \vee (b \vee a) \vee c$

$= a \vee (a \vee b) \vee c$

$= (a \vee a) \vee (b \vee c)$

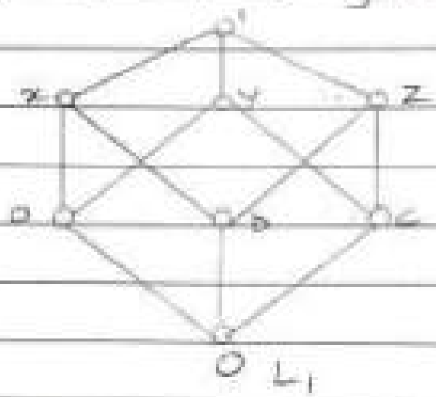
$= a \vee b \vee c$

Here a, b, c are atoms which are join irreducible.

2] In a Boolean lattice every non-zero element is join of atoms.

3] J(L) = collection of all non-zero join irreducible elements, of Lattice L.

1] Find $J(L_1)$ if L_1 is given by.



→ Let,

Here $1 = x \vee y$

$\therefore$ 1 is join reducible.

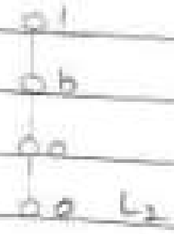
similarly, $x = a \vee b$, $y = a \vee c$, $z = b \vee c$

$\therefore$ x, y, z are join reducible elements.

clearly, a, b, c are join irreducible elements

$\therefore$ $J(L_1) = \{a, b, c\}$

2) Find $J(L_2)$ where,



→ Let,

Here, clearly, 0, a, b, 1 are join irreducible elements

$\therefore J(L_2) = \{a, b, 1\}$

3] Find $J(L_3)$ where,

→ Let, $1 = a \vee b$ $\therefore$ 1 is join reducible.
Here, a, b are join irreducible.
$\therefore J(L_3) = \{a, b\}$.

4] Find $J(L_4)$, where

→ Let, $1 = a \vee b$ or $1 = b \vee c$ $\therefore$ 1 is join reducible
Here, a, b, c are join irreducible.
$\therefore J(L_4) = \{a, b, c\}$.

Note:-

1] $J(L)$ forms a poset under partial ordering of L.

2] $J(L)$ need not be lattice.
e.g. $J(L_1)$, $J(L_3)$, $J(L_4)$ are not lattices, but $J(L_2)$ is lattice.

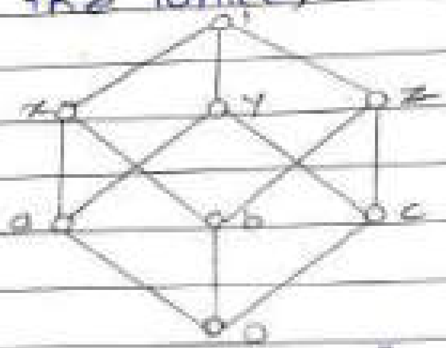
Def^n:-

We define,
$\xi(a) = \{x \in L \mid x \leq a,\ x \in J(L)\}$ for $a \in L$
= Set of all non-zero join irreducible elements which are related to a.

$\xi(a) = (a] \wedge J(L)$

Note:-

1) Consider the lattice,



Here,

$\xi(1) = (1] \wedge \{a, b, c\} = \{a, b, c\}$

$\xi(x) = (x] \wedge \{a, b, c\} = \{a, b\}$

$= \{x, a, b, 0\}$

$\xi(y) = (y] \wedge \{a, b, c\}$ $\{a, c\}$

$\xi(z) = (z] \wedge \{a, b, c\}$ $\{b, c\}$

$\xi(a) = \{a\}$

$\xi(b) = \{b\}$

$\xi(c) = \{c\}$

$\xi(0) = (0] \wedge \{a, b, c\} = \phi$

2) $\xi(a)$ is poset but need not be lattice.

3) $\xi(1) = J(L)$.

4) $\xi(a) = \{a\}$ if a is atom.

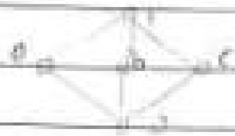
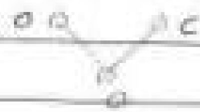
Hereditory subset:-

For a poset P we call $H \subseteq P$ is hereditary subset if $x \in H$, $a \in P$ such that, $a \leq x \Rightarrow a \in H$.

Note:-

1] Every ideal is hereditory subset but converse need not be true.

e.g.

Let L is  its hereditory subset is,  which is not ideal.

2] Every hereditory subset is a poset with ordering relation on P, and need not be lattice.

3] Every hereditory subset of chain is ideal.

4] Intersection of two hereditory subset is hereditory and union of two hereditory subset is also hereditory.

5] collection of hereditory subsets forms a poset under set inclusion.

Th^m 28 :-

H(P) = Collection of all hereditory subsets of poset P. Then show that H(P) is lattice.

→ Proof:-

Let H(P) = collection of all hereditory subset of poset P.

We define meet and join for H(P) as set inclusion.

i.e. Let A, B ∈ H(P) be any elements.

Then $A \wedge B = A \cap B$ and $A \vee B = A \cup B$.

<u>Claim</u>:- $A \wedge B$ and $A \vee B$ are hereditory subsets of P.

Let $x \in A \wedge B$ and $a \in P$ such that $a \leq x$

As $x \in A \wedge B$

$\Rightarrow x \in A \cap B$

$\Rightarrow x \in A$ and $x \in B$ and (with $a \in P$) s.t. $a \leq x$

$\Rightarrow a \in A$ and $a \in B$ ($\because A$ & B are hereditory)

$\Rightarrow a \in A \cap B$

$\Rightarrow a \in A \wedge B$

$\therefore A \wedge B$ is hereditory subset of poset P.

$\Rightarrow A \wedge B \in H(P)$. $\forall A, B \in H(P)$. —— ①

Let $y \in A \vee B$ and $a \in P$ such that $a \leq y$.

As $y \in A \vee B$

$\Rightarrow y \in A \cup B$

$\Rightarrow y \in A$ or $y \in B$ and (with $a \in P$) s.t. $a \leq y$

$\Rightarrow a \in A$ or $y \in B$ ($\because A, B$ are hereditory)

$\Rightarrow a \in A \cup B$

$\Rightarrow a \in A \vee B$

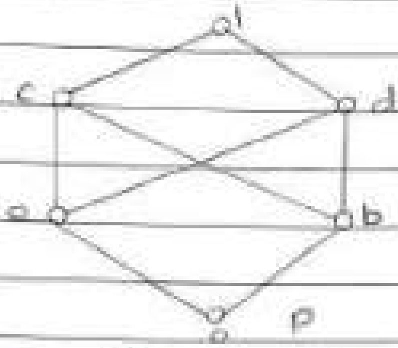
$\therefore A \vee B$ is hereditory subset of poset P.

$\Rightarrow A \vee B \in H(P)$ $\forall A, B \in H(P)$ —— ②

By ① & ②,

$H(P)$ forms a lattice.

1) Consider the poset P defined by Hasse diagram given below.



It's hereditary subsets are,

$H_1 = \{0\}$
$H_2 = \{0, a\}$
$H_3 = \{0, b\}$
$H_4 = \{0, a, b\}$
$H_5 = \{0, a, b, c\}$
$H_6 = \{0, a, b, d\}$
$H_7 = \{0, a, b, c, d\}$
$H_8 = \{0, a, b, c, d, 1\}$

Here,

$$H_1 \subset \begin{matrix} H_2 \\ H_3 \end{matrix} \subset H_4 \subset \begin{matrix} H_5 \\ H_6 \end{matrix} \subset H_7 \subset H_8.$$

$\therefore H(P) = \{H_1, H_2, H_3, H_4, H_5, H_6, H_7, H_8\}$

H(P)

Clearly, H(P) is lattice.

Q. If $A, B, C \in H(P)$ then $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$?
~~Just~~ Justify.

→ Let,
$x \in A \wedge (B \vee C)$ and $a \in P$ such that $a \leq x$

<u>Claim</u> :- $a \in (A \wedge B) \vee (A \wedge C)$

As $x \in A \wedge (B \vee C)$

$\Rightarrow x \in A \cap (B \cup C)$

$\Rightarrow x \in A$ and $x \in B \cup C$

$\Rightarrow x \in A$ and $x \in B$ or $x \in C$

$\Rightarrow x \in A$ and $x \in B$ or $x \in A$ and $x \in C$

$\Rightarrow x \in A \cap B$ or $x \in A \cap C$ with $a \leq x$

$\Rightarrow$ ~~$x \in$~~ $a \in A \cap B$ or $a \in A \cap C$

$\Rightarrow a \in (A \cap B) \cup (A \cap C)$

$\Rightarrow a \in (A \wedge B) \vee (A \wedge C)$

$\therefore A \wedge (B \vee C)$ is Hereditory and subset of $(A \wedge B) \vee (A \wedge C)$ which is Hereditory.

$A \wedge (B \vee C) \subseteq (A \wedge B) \vee (A \wedge C)$.

Simillarly, we can easily prove,

$(A \wedge B) \vee (A \wedge C) \subseteq A \wedge (B \vee C)$

$\therefore (A \wedge B) \vee (A \wedge C) = A \wedge (B \vee C) \quad \forall A, B, C \in H(P)$

$\therefore H(P)$ is distributive lattice.

Th^m 29 :-

If L is finite distributive lattice then the map $\phi : L \rightarrow H(J(L))$ is an isomorphism.

→ <u>Proof</u> :-

Let L be a finite distributive lattice.

Define $\phi : L \rightarrow H(J(L))$ by

$\phi(a) = \downarrow(a)$ (poset)

$\{a\} \wedge J(L)$

i) To prove ϕ is well-defined.

Let $a, b \in L$ be any elements such that

$$a = b$$
$$(a] = (b]$$
$$(a] \wedge J(L) = (b] \wedge J(L)$$
$$\xi(a) = \xi(b)$$
$$\phi(a) = \phi(b)$$

This shows that ϕ is well defined.

ii) To prove ϕ is one-one.

Let $\phi(a) = \phi(b)$

$\Rightarrow \quad \xi(a) = \xi(b)$

$\Rightarrow \quad \vee \xi(a) = \vee \xi(b)$

$\Rightarrow \quad a = b$

This shows that ϕ is one-one.

iii) To prove ϕ is onto.

Let $A \in H(J(L))$ be any element.

Let $\vee A = a$

So $\quad A \leq \xi(a)$ —— (I)

Let $x \in \xi(a)$

$\Rightarrow \quad x \leq a$

$\Rightarrow \quad x = x \wedge a$

$\Rightarrow \quad x = x \wedge (\vee A)$

$\Rightarrow \quad x = x \wedge (y_1 \vee y_2 \vee \cdots \vee y_k)$

$\Rightarrow \quad x = (x \wedge y_1) \vee (x \wedge y_2) \vee (x \wedge y_3) \vee \cdots \vee (x \wedge y_k)$

So $\exists\, y_i \in A$ such that $x = x \wedge y_i$

$\Rightarrow \quad x \leq y_i$

$\Rightarrow \quad x \in A$ $\quad$ ($\because$ A is hereditary subset)

$\xi(a) \leq A$ —— (II)

By (I) & (II), $A = \xi(a) = \phi(a)$

This shows that ϕ is onto.

iv] To prove ϕ is Homomorphism.

Let $x \in \xi(a \vee b) = (a \vee b] \wedge J(L)$

$\Rightarrow x \in (a \vee b]$ & $x \in J(L)$

$\Rightarrow x \leq a \vee b$ & $x \in J(L)$

$\Rightarrow x = x \wedge (a \vee b]$ & $x \in J(L)$

$\Rightarrow x = (x \wedge a) \vee (x \wedge b)$ & $x \in J(L)$

$\Rightarrow x = x \wedge a$ or $x = x \wedge b$ & $x \in J(L)$

$\Rightarrow x \leq a$ or $x \leq b$ & $x \in J(L)$

$\Rightarrow x \in (a]$ or $x \in (b]$ & $x \in J(L)$

$\Rightarrow x \in \xi(a)$ or $x \in \xi(b)$

$\Rightarrow x \in \xi(a) \vee \xi(b)$

$\therefore \xi(a \vee b) \subseteq \xi(a) \vee \xi(b)$ — (iii)

Let $x \in \xi(a) \vee \xi(b)$

$x \in \xi(a)$ or $x \in \xi(b)$

$\Rightarrow x \in (a] \wedge J(L)$ or $x \in (b] \wedge J(L)$

$\Rightarrow x \in (a]$ or $x \in (b]$ & $x \in J(L)$

$\Rightarrow x \leq a$ or $x \leq b$ & $x \in J(L)$

$\Rightarrow x \leq a \vee b$ & $x \in J(L)$

$\Rightarrow x \in (a \vee b]$ & $x \in J(L)$

$\Rightarrow x \in (a \vee b] \wedge J(L)$

$\Rightarrow x \in \xi(a \vee b)$

$\therefore \xi(a) \vee \xi(b) \subseteq \xi(a \vee b)$ — (iv)

$\therefore$ By (iii) & (iv)

$\xi(a) \vee \xi(b) = \xi(a \vee b)$

$\Rightarrow \phi(a) \vee \phi(b) = \phi(a \vee b) \quad \forall a, b \in L$

$\therefore \phi$ is join homomorphism. — (A)

We prove meet homomorphism.

We know that,

$(a \wedge b] = (a] \wedge (b]$

$\phi(a \wedge b) = \xi(a \wedge b)$

$= (a \wedge b] \wedge J(L)$

$= (a] \wedge (b] \wedge J(L) \wedge J(L)$

$= (a] \wedge J(L) \wedge (b] \wedge J(L)$

$= \xi(a) \wedge \xi(b)$

$= \phi(a) \wedge \phi(b)$

$\therefore$ ϕ is meet homomorphism — $**$

$\therefore$ By $(*)$ & $(**)$,

ϕ is homomorphism.

Then by (i) to (iv),

ϕ is isomorphism.

ie $L \cong H(J(L))$

where L is finite distributive lattice.

Ring of sets :-

Let A be any non-empty set & consider $P(A)$ then a subset S of $P(A)$ is called ring of sets if $\forall x, y \in S \Rightarrow x \cup y$ and $x \cap y \in S$.

1) Let $x = \{a, b, c\}$

→ Let,

$P(x) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, x\}$

$S_1 = \{\{a\}, \{b\}\}$

⇒ S_1 is not ring of sets, because union and intersection not exists.

$S_2 = \{\phi, \{a\}, \{b\}\}$

⇒ S_2 is not ring of set.

$S_3 = \{\phi, \{a\}, \{b\}, \{a,b\}\}$

⇒ S_3 is ring of set.

$S_4 = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}\}$

⇒ S_4 is not ring of sets.

$S_5 = \{P(X)\}$

$\Rightarrow S_5$ is ring of sets

$S_6 = \phi$

$\Rightarrow S_6$ is ring of set.

Note:-

1] Ring of sets is sublattice of lattice $L = P(X)$

2] Every ring of a set is modular as well as distributive. (Because lattice of power set of any set is modular and distributive)

3] $J(L)$ is not a ring of sets.

4] Collection of hereditory subsets form a ring of sets.

5] $H(J(L))$ is ring of set.

Corollary :-

A finite lattice is distributive iff it is isomorphic to ring of sets.

→ Proof:-

Let L is finite distributive lattice.

$\Rightarrow L \cong H(J(L))$.

$\Rightarrow L \cong$ Ring of sets.

Conversely,

Let L is isomorphic to ring of sets.

To prove, L is distributive.

By hypothesis it is sufficient to show

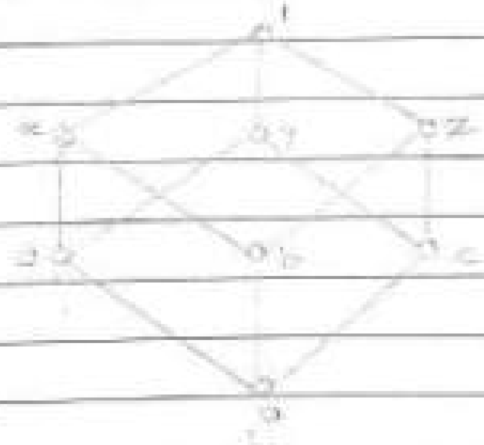
that ring of set is distributive.
As ring of set is subset of P(A) for some non-empty set A and P(A) is distributive.
$\Rightarrow$ Ring of set is distributive.
$\therefore$ L is distributive lattice.

Redundant Representation:-

Let L be a lattice and $a \in L$ be any element with $a = x_1 \vee x_2 \vee \cdots \vee x_n$ is called redundant representation of 'a' if
$a = x_1 \vee x_2 \vee \cdots \vee x_{i-1} \vee x_{i+1} \vee \cdots \vee x_n$ (x_i remove)
for some i.

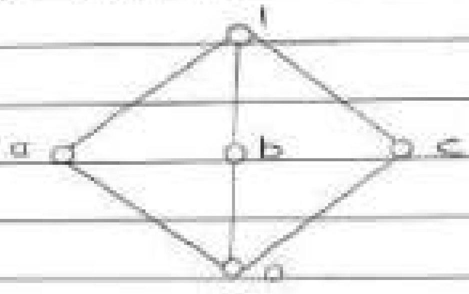
Otherwise such a representation is called irredundant representation.

1) consider the lattice,



$\rightarrow$ Let,
Here, $1 = x \vee y \vee z$ is redundant representation
Because $1 = x \vee y$ or $1 = x \vee z$ or $1 = y \vee z$
Now, $1 = a \vee b \vee c$ is irredundant representation.
Because $1 \neq a \vee b$, $1 \neq b \vee c$ or $1 \neq a \vee c$.

2) Consider the lattice.



M_3

→ Let,

Here $1 = a \vee b \vee c$

Clearly, this representation is redundant

Because $1 = a \vee b$ or $1 = b \vee c$ or $1 = a \vee c$

Also, $a \vee b \vee 0 = 1$ is redundant representation

Because $1 = a \vee b$.

Corollary :-

Every maximal chain C of finite distributive lattice is of length $|J(L)|$.

→ Proof:-

Let L is finite distributive lattice.

Let C is maximal chain of L.

$J(L)$ = Collection of non-zero join irreducible elements of L.

Let $m(a)$ denote the smallest member of chain C containing 'a'.

We know that, Every element of finite distributive lattice have unique irredundant representation. (Join of join irreducible elements)

Now we define,

$\phi : J(L) \to C$ by,

$\phi(a) = m(a) \quad \forall a \in J(L)$

We have $a \le m(a)$

claim :- ϕ is one-one and onto non-zero elements of C.

i.e. any element of $J(L)$ not map to zero

Clearly, ϕ is well-defined.

I) To prove ϕ is one-one:

Let $m(a) = m(b) \quad \forall m(a), m(b) \in C$

and $x \prec m(a)$ & $x \prec m(b)$

for some $x \in C$

Claim:- $x \vee a = x \vee b$

As $x \le x \vee a \le m(a)$, $x \le x \vee b \le m(b)$

if $x = x \vee a$

$\Rightarrow \quad a \leq x$

$\Rightarrow$ x is smallest element containing 'a' which is contradiction.

$\therefore \quad x \vee a = m(a)$

Similarly, $\quad x \vee b = m(b)$

But $\quad m(a) = m(b)$

$\Rightarrow \quad x \vee a = x \vee b$

Let,

$a = a \wedge (x \vee a)$ $\quad$ ∵ (Absorption)

$= a \wedge (x \vee b)$

$= (a \wedge x) \vee (a \wedge b)$ $\quad$ (∵ Distributive)

$\Rightarrow$ Either $a = a \wedge b$ or $a = a \wedge x$ $\quad$ (∵ $a \in J(L)$)

$\Rightarrow \quad a \leq b$ or $a \leq x$

which is not true.

$a \leq b$ $\quad$ —— ①

Now,

$b = b \wedge (x \vee b)$

$= b \wedge (x \vee a)$

$= (b \wedge x) \vee (b \wedge a)$

$\Rightarrow$ Either $b = b \wedge x$ or $b = b \wedge a$

$\Rightarrow \quad b \leq x$ or $b \leq a$

But $b \leq x$ is not true

$\therefore \quad b \leq a$ $\quad$ —— ②

$\therefore$ From ① & ②,

$a = b$

$\therefore \quad m(a) = m(b)$

$\Rightarrow \quad a = b$

$\therefore \quad \phi$ is one-one.

•] To prove ϕ is onto.

Let $y, z \in C$ with $y \prec z$

$\Rightarrow \quad \xi(y) \subset \xi(z)$

$\Rightarrow \quad \exists \; a \in \xi(z) - \xi(y)$ or $a \in \xi(z) \setminus \xi(y) \subseteq J(L$

with $m(a) = z$ $\qquad$ $(\because a \in \xi(z)$ & $a \notin \xi(y))$

$\Rightarrow \quad \phi(a) = z$

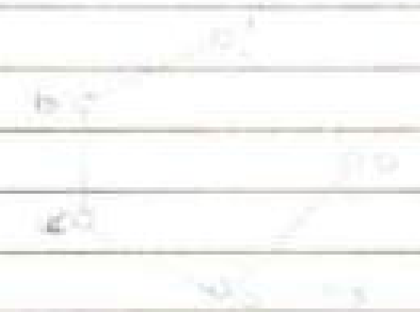
$\Rightarrow \quad \phi$ is onto.

By ① & ②, ϕ is one-one and onto

i.e. $\ell(C) = |J(L)|$

Note:-

1] Converse of above corollary need not be true for eg.



Here. $J(L) = \{a, b, c\}$

$|J(L)| = 3$

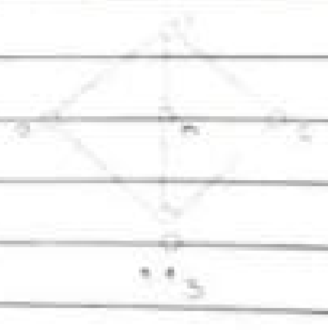
$C = 0 \prec c \prec b \prec 1$

$\ell(C) = 4 - 1 = 3$

$\therefore |J(L)| = \ell(C)$

But N_5 is not distributive lattice.

2]



Stone's Th^m / stone's separation Th^m 30:-

Let L be a distributive lattice, I - ideal of L, D - Dual of ideal of L with $I \cap D = \phi$ then there exist a prime ideal P in L such that $I \subseteq P$ & $P \cap D = \phi$.

→ Proof:-

Let 'L' be a distributive lattice.

Let 'I' is ideal of L and D is dual ideal of L with $I \cap D = \phi$.

Now we construct,

$X = \{ J \subseteq L \mid I \subseteq J, J \cap D = \phi \}$

where J is ideal of L.

clearly $I \in X$.

⇒ X is a non-empty set.

Let C be chain in X.

Let $M = \cup \{ A \mid A \in C \}$

Claim :- $M \in X$

Let $x, y \in M$ be any elements.

To prove $x \wedge y$ & $x \vee y \in M$.

By def^n of M $\exists A_1, A_2 \in C$ such that $x \in A_1$ and $y \in A_2$.

[clearly $A_1, A_2 \subseteq M$ also].

But we have,

either $A_1 \subset A_2$ or $A_2 \subset A_1$

Bcoz $A_1, A_2 \in C$ which is chain.

Let $A_1 \subset A_2$

$\Rightarrow x, y \in A_2$

$\Rightarrow x \wedge y, x \vee y \in A_2$ ($\because A_2$ is ideal)

$\Rightarrow x \wedge y, x \vee y \in M$

Now, $t \leq m$ for $m \in M$

As $m \in M$

$\Rightarrow$ $m \in A_k$ for some k

As A_k is ideal then

$t \leq m \Rightarrow t \in A_k$

$\Rightarrow t \in M$

$\therefore$ M is ideal and clearly $I \subseteq M$.

Now we claim that, $M \cap D = \phi$.

Let $M \cap D \neq \phi$

$\Rightarrow$ $M \cap D$ = some non-empty set

$\Rightarrow$ $\exists\, x \in M \cap D$

$\Rightarrow$ $x \in M$ and $x \in D$

$\Rightarrow$ $x \in A_k$ for some $A_k \in C$ and $x \in D$

$\Rightarrow$ $x \in A_k \cap D$

which is contradicts to definition of X.

$\therefore$ We must have $M \cap D = \phi$.

$\Rightarrow$ $M \in X$ and M is upper bound for C.

And this is true for every chain in X.

Then by Zorn's Lemma X has maximal element say P.

ie. P is an ideal it contains I and $P \cap D = \phi$.

<u>claim:-</u> P is prime ideal.

Let P is not prime ideal.

$\Rightarrow$ $a \wedge b \in P$ but $a \notin P$, $b \notin P$

$\Rightarrow$ $P \subseteq P \vee (a]$ & $P \subseteq P \vee (b]$.

By maximality of P we have,

$(P \vee (a]) \cap D \neq \phi$

and $(P \vee (b]) \cap D \neq \phi$.

$\Rightarrow \exists\ x \in (P(a]) \cap D$

$\Rightarrow \quad x \in P \vee (a]$ & $x \in D$

$\Rightarrow \quad x = p \vee a$ for some $p \in P$ and $a \in (a]$

Similarly,

$\exists\ y \in (P \vee (b]) \cap D$

$\Rightarrow \quad y \in P \vee (b]$ and $y \in D$

$\Rightarrow \quad y = q \vee b$ for some $q \in P$ and $b \in (b]$.

As $x, y \in D$.

$\Rightarrow \quad p \vee a$ and $q \vee b \in D$

$\Rightarrow \quad (p \vee a) \wedge (q \vee b) \in D$ ($\because$ D is ideal)

Now, $(p \vee a) \wedge (q \vee b)$

$= [(p \vee a) \wedge q] \vee [(p \vee a) \wedge b]$

$= \{(p \wedge q) \vee (a \wedge q)\} \vee \{(p \wedge b) \vee (a \wedge b)\} \in P$

($\because$ $p \wedge q \in P$, $a \wedge q \in P$, P is an ideal
$p \wedge b \in P$ & $a \wedge b \in P$ assumptio)

$\Rightarrow \quad (p \vee a) \wedge (q \vee b) \in P \cap D$

But $P \cap D = \phi$

P is not prime.

$\Rightarrow$ P is prime ideal.

$\therefore$ We have a prime ideal P which covers I and $P \cap D = \phi$.

Hence the result.

Corollary:-

Let L be a distributive lattice and I be an ideal for $a \in L$, $a \notin I$ then $\exists$ prime ideal P such that $I \subseteq P$ and $a \notin P$.

$\rightarrow$ Proof:-

consider $I = (b]$ and its dual ideal $D = [b)$

$\Rightarrow I \cap D = \phi$ with $a \notin I$

Then by Stone's Th^m,

$\exists$ prime ideal P such that $I \subseteq P$ and $P \cap D = \phi$

If $a \in P$ then $P \cap D \neq \phi$.

which is contradiction.

$\therefore a \notin P$

Hence the result.

Corollary :-

If L is distributive lattice. Let $a, b \in L$ such that $a \neq b$ then $\exists$ a prime ideal containing exactly one of a & b.

→ Proof:-

Let $a, b \in L$ such that, $a \neq b$

1) Let $a < b$

Let $I = (a]$ and $D = [a)$

$\Rightarrow I \cap D = \phi$

Then by Stone's th^m $\exists$ prime ideal P such that.

$I \subseteq P$ and $P \cap D = \phi$.

As $b \notin I \Rightarrow b \notin P$

But $I \subseteq P \Rightarrow a \in P$

2) Let $b < a$

By simillar argument we can prove.

$a \notin P$ and $b \in P$.

By ① & ② we get required result.

Corollary :-

Every ideal I of distributive lattice is the intersection of all prime ideals containing it.

→ **Proof:-**

Let $I' = \cap \{P \mid P \text{ is prime ideal}, I \subseteq P\}$

Claim:- $I = I'$

Clearly, $I \subseteq I'$ — ①

Now we will show that,

$I' \subseteq I$

Let $I' \nsubseteq I$

$\Rightarrow a \in I' \setminus I$

$\Rightarrow a \in I'$ and $a \notin I$

We know that L is distributive lattice and I be an ideal such that $a \in L$ and $a \notin I$ then $\exists$ a prime ideal P such that $I \subseteq P$ and $a \notin P$.

By defn of I' we get,

$I' \subseteq P$, and $a \notin P$,

$\Rightarrow a \notin I'$

which is contradiction.

Hence we must have, $I' \subseteq I$ — ②

$\therefore$ From ① + ②,

$I = I'$.

Hence the result.

Field of sets:-

A ring of sets is called field of sets if it is closed under set complementation. (Complement of each element exists + belongs to set.).

Note:-

1) Field of sets is also called Stone's Lattice.

2) Every field of sets is ring of sets but converse need not be true.

eg

Let $x = \{a, b, c\}$

$R_1 = \phi$, $R_2 = \{\phi, \{a\}, \{b, c\}, x\}$

$R_3 = \{\phi, x\}$

$R_4 = \{\phi, \{b\}, \{b, c\}\}$

$R_5 = \{\{a\}\}$

$R_6 = \{P(x)\}$

→ Let,

R_1 is not field of set

R_2 is field of set

R_3 is field of set

R_4 is not field of set

R_5 is not field of set

R_6 is field of set

Thm 31:-

L be a distributive lattice with 0 and 1, $0 \neq 1$ then L is Boolean lattice iff $P(L)$ is unordered where $P(L)$ is collection of all prime ideals of a lattice L.

→ Proof:-

Let L be a distributive lattice with 0 and 1 also $0 \neq 1$.

Suppose that L is Boolean lattice.

Claim:- $P(L)$ is unordered.

Let $A, B \in P(L)$ with $A \subseteq B$ and $A \neq B$.

Let $a \in B - A$.

Since L is Boolean, complement of a exist

say a'.
Since $a \vee a' = 1$
We have $a' \notin B$ otherwise $a \vee a' = 1 \in B$
($\because a \in B, a' \in B \Rightarrow a \vee a' = 1 \in B$)

$\Rightarrow B = L$
The prime we have proper ideal.
This is not possible because B is proper ideal of L.
Since $a \in A$, $a' \notin A$ $\quad \because A \subset B, a' \notin B$
but $a \wedge a' = 0 \in A$
which is contradiction to primeness of A.
$\therefore$ We must have.
$$A = B$$
Every prime ideal in a Boolean lattice, is a maximal ideal.
$\therefore$ P(L) is unordered.

Conversely,
Suppose P(L) is unordered.
Claim:- L is Boolean.
It is sufficient to show that L is complemented.
on contrary suppose $a \in L$ has no complement.
Consider,
$$D = \{x \in L \mid x \vee a = 1\}$$
We claim that D is dual ideal.
As $1 \vee a = 1 \in D$
$\therefore$ D is non-empty.
Let $x, y \in D$
$\therefore x \vee a = 1 = y \vee a$

$(x \wedge y) \vee a = (x \vee a) \wedge (y \vee a)$
$= 1 \wedge 1$
$= 1$
$\Rightarrow x \wedge y \in D$
$(x \vee y) \vee a = x \vee (y \vee a)$
$= x \vee 1$
$= 1$
$\Rightarrow x \vee y \in D$.

For $z \in L$ with $x \leq z$, $x \in D$

We have,
$x \vee a \leq z \vee a$
$1 \leq z \vee a$
$\Rightarrow 1 = z \vee a$
$\Rightarrow z \in D$

$\therefore$ D is Dual ideal.

Consider,
$D_1 = D \vee [a)$

It is clear that D_1 is also dual ideal.
(In distributive lattice join of two ideal is an ideal).

Clearly, $0 \notin D_1$

If so $0 \in D_1$
$\Rightarrow \exists\ b \in D_1$ (or D) s.t. $a \wedge b = 0$ ($\because$ Distributivity)
and also $a \vee b = 1$.
$\therefore$ b is complement of a.
$\therefore$ We must have $0 \notin D$.

The set generated by '0' is an ideal (i.e. $(0]$.)
$\therefore (0] \wedge D_1 = \phi$.
$\therefore \exists$ prime P_1 such that $(0] \subseteq P_1$ & $P_1 \wedge D_1 = \phi$
($\because$ By Stone's th^m).

Now consider,

$(0] \cap P_1$ it is also an ideal.

and $1 \notin (0] \vee P_1$ (o.w. $1 \in P_1 \Rightarrow P_1 = L$ not prime)

ie o.w. $1 \in P_1$ $\because 1 \neq (0]$

$\Rightarrow$ $P_1 = L$ and P_1 is not prime

$\therefore$ $1 \notin (0] \vee P_1$

$\therefore$ $(0] \vee P_1 \cap [1) = \phi$.

By stone's separation th^m,

$\exists$ prime ideal P_2 such that,

$(0] \vee P_1 \subseteq P_2$ and $P_2 \cap [1) = \phi$

$\Rightarrow$ $P_1 \leq (0] \vee P_1 \subseteq P_2$

$\Rightarrow$ $P_1 \subseteq P_2$.

which is contradiction to $P(L)$ is unordered.

$\therefore$ We must have complement of every element in L.

Thus L is complemented distributive lattice

Hence a Boolean.

Distributive, Standard and Neutral element:-

1] An element 'a' is called distributive iff

$a \vee (y \wedge z) = (a \vee y) \wedge (a \vee z)$ $\forall\ y, z \in L$.

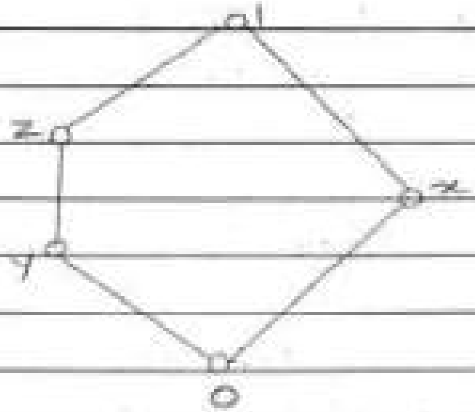
2] An element 'a' is called standard iff

$x \wedge (a \vee z) = (x \wedge a) \vee (x \wedge z)$ $\forall\ x, z \in L$.

3] An element 'a' is called neutral iff

$(a \wedge y) \vee (y \wedge z) \vee (z \wedge a) = (a \vee y) \wedge (y \vee z) \wedge (z \vee a)$.

e.g.



N_5

Distributive element

1] $a = 0$, $y, x \in N_5$

$LHS = 0 \vee 0 = 0$

$RHS = (0 \vee y) \wedge (0 \vee x)$

$= y \wedge x$

$= 0$

$\Rightarrow$ 0 is distributive.

simillarly, 1 is distributive.

2] $a = y$, $0, y \in N_5$

$LHS = y \vee (0 \wedge z) = y$

$RHS = (y \vee 0) \wedge (y \vee z)$

$= y \vee z$

$= y$

$LHS = y \vee (z \wedge x) = y$

$RHS = (y \vee z) \wedge (y \vee x)$

$= z \wedge 1$

$= z$

$\Rightarrow$ y is not distributive.

3) $a = x$, $y, z \in N_5$

$LHS = x \vee (y \wedge z) = x \vee y = 1$

$RHS = (x \vee y) \wedge (x \vee z)$

$= 1 \wedge 1$

$= 1$

$\therefore$ x is distributive.

4) $a = z$, $x, y \in N_5$

$LHS = z \vee (y \wedge x) = z \vee 0 = z$

$RHS = (z \vee y) \wedge (z \vee x)$

$= z \wedge 1$

$= z$

$\therefore$ z is distributive.

Standard elements (If distributive then check for standard).