Euler's Lag. eq^n

Variational Principles

Introduction:-

The piecewise information of the path $y = f(x)$ whether it is minimum or maximum at a point can be obtained from differential calculus by solving $f'(x) = 0$. However if we want to know about the whole path we use integral calculus. i.e. the techniques of calculation of variations which are called variational principles.

f is maxi. or min. $f'(x) = 0$

The history of calculus of variation can be traced back to the year 1696 when John Bernoulli advanced the Branchistrochrone problem (Greek word Branchistos = shortest chronos = time). In this problem one has to find the curve connecting to two given points A and B, that do not lie in vertical such that a particle sliding down this curve under gravity. ~~fo~~ from A reaches to point B in shortest time.

$f'(x) = 0$, $f''(x) > 0$ ↓ minimum; if $f''(x) < 0$ it is maximum

B shortest distance complete in shortest time

Simillar problems were,

i] Problem of geodesic.
ii] Problem of minimum surface of revolution.
iii] Isopemetric problem.

Note:-

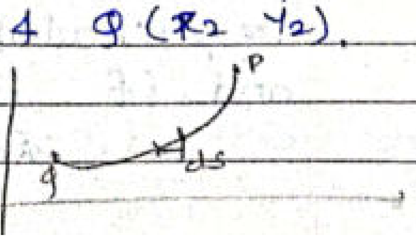
Extremization (either minimum or maximum) of length of curve.

→ shortest distance

Consider the curve $y = f(x)$ in a plane joining the points $P(x_1, y_1)$ & $Q(x_2, y_2)$.

Consider infinitesimal distance bet^n 2 points (very small).

ds dy



on this curve,

$$ds = \sqrt{dx^2 + dy^2}$$

Ingeneral,

The length of total path, is,

$$I(y) = \int_P^Q ds$$

$\sqrt{x} = \frac{1}{2\sqrt{x}}$

$$= \int_P^Q \sqrt{dx^2 + dy^2}$$

$$= \int_P^Q \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$I(y) = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, dx \quad \text{——— ①}$$

We have to find a curve y such that eqⁿ ① is extremum.

y is funⁿ of x & y' is funⁿ of y.

Ingeneral we have to extremize

$$I(y) = \int_{x_1}^{x_2} f(x, y, y')\, dx$$

In this case I is called functional.

[Functional means a function whose argument is again a argument].

Here I is function of y & y is a funⁿ of x. ∴ I is functional.

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State and prove basic lemma ~~of~~ variational calculus

Statement :-

If x_1 & x_2 are fixed constants and $G(x)$ is particular continuous function for $x_1 \le x \le x_2$ and if $\int_{x_1}^{x_2} G(x) \cdot \eta(x)\, dx = 0$ for every choice of continuous differentiable function $\eta(x)$ such that $\eta(x_1) = \eta(x_2) = 0$ then $G(x) = 0$ identically in $x_1 \le x \le x_2$.

<u>Proof:-</u>

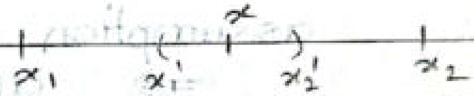
On contrary, suppose Lemma is not true.

$G(x')\neq 0$ i.e. it is either $G(x')>0$ or $G(x')<0$

Let us assume that there is a particular value x' in interval such that $G(x') \neq 0$

Let us assume $G(x') > 0$

Since $G(x)$ is continuous in $x_1 \le x \le x_2$. So it is continuous at x'. Hence there exists interval $x_1' \le x \le x_2'$ in which $G(x) > 0$, everywhere.



Define,
$$\eta(x) = \begin{cases} 0 & \text{if } x_1 \le x \le x_1' \\ (x-x_1')^2 \cdot (x-x_2')^2 & \text{if } x_1' \le x \le x_2' \\ 0 & \text{if } x_2' \le x \le x_2 \end{cases}$$

for this choice of $\eta(x)$ which also satisfies $\eta(x_1') = 0 = \eta(x_2')$

Clearly, η is continuous on $[x_1, x_2]$

$\therefore$ We have,
$$\int_{x_1}^{x_2} G(x)\cdot \eta(x)\, dx$$

$$= \int_{x_1}^{x_1'} G(x)\cdot\eta(x)\, dx + \int_{x_1'}^{x_2'} G(x)\cdot\eta(x)\, dx + \int_{x_2'}^{x_2} G(x)\cdot\eta(x)\, dx$$

$$= 0 + (x-x_1')^2\cdot(x-x_2')^2 + 0$$
$$= (x-x_1')^2 \cdot (x-x_2')^2$$

$$\therefore \int_{x_1}^{x_2} G(x)\cdot \eta(x)\, dx = \int_{x_1'}^{x_2'} G(x)\cdot (x-x_1')^2 (x-x_2')^2\, dx \qquad \text{--- (2)}$$

$\int \underbrace{c_1 c_2 c_3}_{\text{dx} > 0} \, dx = \text{positive}$

Since $G(x) > 0$ in $x_1' \le x \le x_2'$

$\therefore$ RHS of eq^n ② is definitely positive

$c_1 = G(x)$
$c_2 = (x - x_1')$
$c_3 = (x - x_2')$

$$\therefore \int_{x_1}^{x_2} G(x) \cdot \eta(x) \, dx > 0$$

This contradicts to the hypothesis

$$\int_{x_1}^{x_2} G(x) \cdot \eta(x) \, dx = 0.$$

Simillarly, if $G(x) < 0$, we obtain simillar contradiction. This contradiction is due to assumption $G(x) \neq 0$ for x in $[x_1, x_2]$

$\Rightarrow G(x) = 0$ identically in $[x_1, x_2]$

Hence the proof.

1] Find the Euler-Lagrange's d.e. satisfied by twice differentiable fun^n $y(x)$ which extremizes the functional $I(y(x)) = \int_{x_1}^{x_2} f(x, y, y') \, dx$ where y' is prescribed at the end points i.e. end points of the curve $y(x)$ are fixed.

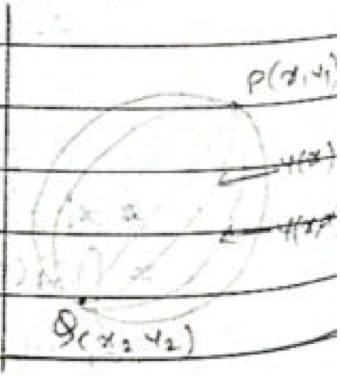
→ consider,

$P(x_1, y_1)$ & $Q(x_2, y_2)$ be two points in XY plane. We can join P & Q by so many curves.

Let, $y(x)$ be curve joining P & Q which gives extremum value or extremizes the functional

$$I(y) = \int_{x_1}^{x_2} f(x, y, y') \, dx \quad \text{—①}$$

If we change x then the curves are change



Define a family $Y(x; \alpha) = y(x) + \alpha \eta(x)$

of neighbouring curves y such that η is c' function ~~wh~~ with $\eta(x_1) = \eta(x_2) = 0$ — ③

& α is parameter.

If we change α we get the different curves.

Note that, $y(x_1, \alpha) = y(x_1) + \alpha \cdot \eta(x_1)$
$= y(x_1)$ $\because \eta(x_1) = 0$ — by ③

$y(x_2, \alpha) = y(x_2) + \alpha \cdot \eta(x_2)$
$= y(x_2)$ $\because \eta(x_2) = 0$ — by ③

The value of integral ① on family ② is given by,

$$J(\alpha) = \int_{x_1}^{x_2} f(\underset{\text{indpt.}}{x}, y(x, \alpha), y'(x, \alpha))\, dx \quad \text{— ④}$$

For $\alpha = 0$,

The family ② coincides with the curve $y(x)$ which gives extremum value of functional I.

f is extremum when $f'(x) = 0$

$$\therefore \left(\frac{\partial I}{\partial \alpha}\right)\Big|_{\alpha = 0} = 0$$

differential and integral sign

$$\therefore \quad 0 = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \cdot \frac{\partial y(x, \alpha)}{\partial \alpha} + \frac{\partial f}{\partial y'} \cdot \frac{\partial y'(x, \alpha)}{\partial \alpha}\right] dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \cdot \eta(x) + \frac{\partial f}{\partial y'}\, \eta'(x)\right] dx$$

integration by parts rule

$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial y} \cdot \eta(x) + \underbrace{\left[\frac{\partial f}{\partial y'}\, \eta(x)\right]_{x_1}^{x_2}}_{= 0} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \cdot \eta(x)$$

$\because \eta(x_2) = \eta(x_1) = 0$
$\partial y'$
$\eta(x_2) = 0 = \eta(x_1)$

$$0 = \int_{x_1}^{x_2} \frac{\partial f}{\partial y} \cdot \eta(x) - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \cdot \eta(x)\, dx + c$$

$y(x, \alpha) = y(x) + \alpha \cdot \eta(x)$
$= 0 + \left(\frac{\partial \alpha}{\partial \alpha} \cdot \eta(x) + \alpha \frac{\partial \eta(x)}{\partial \alpha}\right) = 1 \cdot \eta(x) + 0$

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$$= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] \eta(x)\, dx \neq 0$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] \eta(x)\, dx$$

By basic Lemma of variational principle,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$-\int_{x_1} G(x)\cdot\eta(x)dx$

$\Rightarrow G(x)=0$ identical

$\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$

This is Euler Lagrange's eq^n.

2] Derive Euler Lagrangian eq^n satisfied by twice differentiable fun^n $y_1, y_2, \ldots y_n$ that extremizes the integral $I = \int_{x_1}^{x_2} f(x, y_1, y_2, y \ldots y_n, y_1', y_2', \ldots y_n')$

where y_i' are prescribed at the end points

→ Proof:-

We write, $I = \int_{x_1}^{x_2} f(x, y_i, y_i')\, dx$

Further we define neighbouring family.

$y_i(x, \alpha) = y_i(x) + \alpha\, \eta_i(x)$

& use the last result,

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = 0 \qquad \forall\, i = 1, 2, \ldots n$$

is, Euler Langrange's eq^n.

Geodesic :-

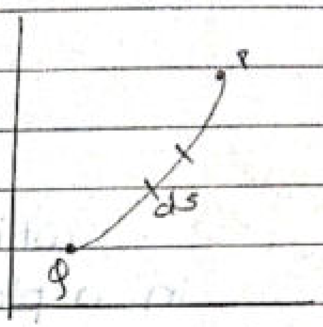
Geodesic is the curve bet^n 2 given points on given surface having extremum lengths (maximum or minimum).

1) Show that geodesic in a euclidean plane is straight line.

→ Let,

$P(x_1, y_1)$ & $Q(x_2, y_2)$ be two points in $\mathbb{R}^2$.

If $y = y(x)$ is curve joining P & Q, ds is arbitrary distance element on y then



$$ds = \sqrt{dx^2 + dy^2}$$

Then length of this curve is given by,

$$I = \int_P^Q ds = \int_P^Q \sqrt{dx^2+dy^2}$$

$$= \int_{x_1}^{x_2} \sqrt{1+(y')^2}\, dx$$

$$\therefore I = \int_{x_1}^{x_2} f(x, y, y')\, dx$$

where $f(x, y, y') = \sqrt{1+(y')^2}$

Here I is extremum if f satisfies Euler Lagrange's eq^n i.e.

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Now, $\frac{\partial f}{\partial y} = 0$ & $\frac{\partial f}{\partial y'} = \frac{1}{2\sqrt{1+(y')^2}} \cdot 2y'$

$$= \frac{y'}{\sqrt{1+(y')^2}}$$

$\frac{(y')^2}{\sqrt{1+(y')^2}}$

$\therefore$ Eq^n becomes,

$$0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = 0$$

$$\Rightarrow \frac{y'}{\sqrt{1+(y')^2}} = \text{constant} = c_1$$

$$\therefore (y')^2 = c_1^2 \cdot [1+(y')^2]$$

$$(y')^2(1-c_1^2)=c_1^2$$

$$\therefore (y')^2 = \frac{c_1^2}{1-c_1^2} \Rightarrow y' = \frac{c_1}{\sqrt{1-c_1^2}} = c_2$$

$$\therefore y' = c \longrightarrow \text{on integrating}$$

$$\therefore y = c_2 x + c_3$$

$ax+by=c$

$c_2x+y=c_3$

which is a straight line. Thus a geodesic in a plane is a straight line.

3] Show that the geodesic on surface of sphere is a arc of the great circle.

→ Let,

Consider, a sphere of radius 'r' described by,

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r\cos\theta$$

Let $P(\theta_1, \phi_1)$ & $Q(\theta_2, \phi_2)$ be two points on sphere.

If we consider two points arbitrary close on curve is given by,

$$ds^2 = dx^2 + dy^2 + dz^2$$
$$= r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

∴ The total length of curve joining P & Q is

$$s = \int_P^Q ds = \int_P^Q r\sqrt{d\theta^2 + \sin^2\theta \, d\phi^2}$$

$dx^2 = \cancel{dr^2 + r^2} \sin$ $dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$

We write, $\phi = \phi(\theta)$ & get following integration

$$s = r\int_P^Q \sqrt{1+\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}\, d\theta$$

$$= \int_{\theta_1}^{\theta_2} r\sqrt{1+\sin^2\theta\ \phi'^2}\, d\theta \qquad \text{where } \phi' = \frac{d\phi}{d\theta}$$

$$= r\int_{\theta_1}^{\theta_2} f(\theta, \phi, \phi')\, d\theta$$

where,

$$f(\theta, \phi, \phi') = \sqrt{1+\sin^2\theta\ \phi'^2}$$

Now the curve has minimum value if f satisfies Euler's Lagrange's eq^n,

$$\frac{\partial f}{\partial \phi} - \frac{d}{d\theta}\left(\frac{\partial f}{\partial \phi'}\right) = 0$$

$$\therefore \frac{\partial f}{\partial \phi} = 0 \quad \& \quad \frac{d}{d\theta}\left(\frac{\partial f}{\partial \phi'}\right) = \frac{1}{2\sqrt{1+\sin^2\theta\cdot\phi'^2}} \times \sin^2\theta\, 2\phi'$$

$$\therefore \quad 0 - \frac{d}{d\theta}\left(\frac{\sin^2\theta\cdot 2\phi'}{2\sqrt{1+\sin^2\theta\cdot\phi'^2}}\right) = 0$$

$$\frac{d}{d\theta}\left[\frac{\sin^2\theta\cdot\phi'}{\sqrt{1+\sin^2\theta\cdot\phi'^2}}\right] = 0$$

$$\therefore \frac{\sin^2\theta\cdot\phi'}{\sqrt{1+\sin^2\theta\,\phi'^2}} = \text{constant} = c_1$$

$\therefore$ on squaring,

$$\sin^4\theta\cdot\phi'^2 = c_1^2(1+\sin^2\theta\cdot\phi_1'^2)$$

$$\frac{1}{c_1^2} = \frac{1+\sin^2\theta\cdot\phi_1'^2}{\sin^4\theta\cdot\phi'^2}$$

$$\frac{1}{c_1^2} = \frac{1}{\sin^4\theta\cdot\phi'^2} + \frac{1}{\sin^2\theta}$$

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$$\frac{1}{c_1^2} - \frac{1}{\sin^2\theta} = \frac{1}{\sin^4\theta \cdot \phi'^2}$$

$$\frac{\sin^2\theta - c_1^2}{c_1^2 \sin^2\theta} = \frac{1}{\sin^4\theta \cdot \phi'^2}$$

$$\therefore \quad \phi'^2 = \frac{c_1^2}{\sin^2\theta[\sin^2\theta - c_1^2]}$$

$\frac{1}{\sin\theta} = \operatorname{cosec}\theta$

$$= \frac{c_1^2 \operatorname{cosec}^2\theta}{[1 - c_1^2 \operatorname{cosec}^2\theta] \sin^2\theta}$$

common out $\sin^2\theta$

$$= \frac{c_1^2 \operatorname{cosec}^4\theta}{[1 - c_1^2(1+\cot^2\theta)]}$$

$\frac{1}{\sin^2\theta} = \operatorname{cosec}^2\theta$

$\operatorname{cosec}^2\theta = 1+\cot^2\theta$

$$= \frac{c_1^2 \operatorname{cosec}^4\theta}{1 - c_1^2 - c_1^2 \cot^2\theta}$$

$$\therefore \quad \phi'^2 = \frac{c_1^2 \operatorname{cosec}^4\theta}{(1-c_1^2)\left[1 - \frac{c_1^2}{1-c_1^2}\cot^2\theta\right]}$$

$$\phi' = \frac{c_1 \operatorname{cosec}^2\theta}{\sqrt{1-c_1^2}\sqrt{1 - \frac{c_1^2}{1-c_1^2}\cot^2\theta}}$$

$\sqrt{1-c_1^2}$

$= \sqrt{\frac{c_1^2}{1-c_1^2}}$

$= k$

$$\therefore \quad \phi' = \frac{k \operatorname{cosec}^2\theta}{\sqrt{1 - k^2\cot^2\theta}} \quad \text{--- (1)}$$

Put, $k \cot\theta = t$

$-k \operatorname{cosec}^2\theta \, d\theta = dt$

By eq^n (1),

$$\phi' = d\phi = \frac{-dt}{\sqrt{1-t^2}}$$

$\cot\theta = -\operatorname{cosec}^2\theta \, d\theta$

on integrating,

$$\phi = -\sin^{-1} t + c_2$$

$$\Rightarrow \sin^{-1} t = c_2 - \phi$$

$$\Rightarrow t = \sin(c_2 - \phi)$$

$$\therefore k\cot\theta = \sin(c_2 - \phi)$$

$$k\cot\theta = \sin c_2 \cdot \cos\phi - \cos c_2 \cdot \sin\phi$$

$$k\frac{\cos\theta}{\sin\theta} = \sin c_2 \cdot \cos\phi - \cos c_2 \cdot \sin\phi$$

multiplying by $\sin\theta$ on both side

$$k\cos\theta = \sin c_2 \sin\theta \cdot \cos\phi - \cos c_2 \cdot \sin\theta \cdot \sin\phi$$

$$kr\cos\theta = r\sin\theta\cos\phi \cdot \sin c_2 - r\sin\theta \cdot \sin\phi \cdot \cos c_2$$

$$kz = \sin c_2\, x - \cos c_2\, y$$

$$kz = ax + by$$

where, $a = \sin c_2$ & $b = -\cos c_2$

radius of circle = radius of sphere ⇒ It is great circle.

This gives eq^n of plane passing through origin 'O' (i.e. center of sphere). The intersection of this plane with sphere is the great circle. Hence,

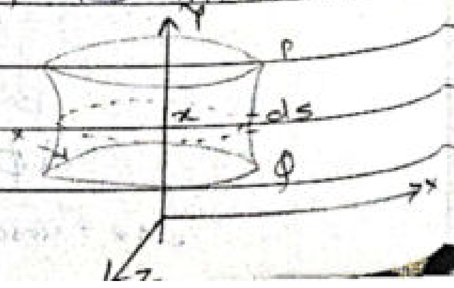
Geodesic on surface of sphere arc of great circle.

4] Find the curve which gives minimum surface of revolution when revolved about Y-axis. OR

Show that the curve is catenary for which area of surface of revolution is minimum when revolved about Y-axis.

→ Let,

Consider a curve joining



points P & Q in XY plane. If we revolve this curve about Y- axis then we get a surface in $\mathbb{R}^2$ as shown in fig.

consider a small length element,

$$ds = \sqrt{dx^2 + dy^2}$$

$$= \sqrt{1+y'^2}\ dx$$

on this curve.

This element ds will form a cylinder during rotation of radius r.

The surface area of this small strip is $2\pi x\, ds$.

The total area is given by,

$$A = \int_P^Q 2\pi x\ ds$$

$$= \int_P^Q 2\pi x \sqrt{1+y'^2}\ dx$$

$$A(y) = \int_{x_1}^{x_2} 2\pi x \sqrt{1+y'^2}$$

$$= \int_{x_1}^{x_2} f(x, y, y')\ dx$$

where, $f = 2\pi x \sqrt{1+y'^2}$

The area A(y) will be minimum if f satisfies Euler's Langrange's eq^n,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$0 - \frac{d}{dx}\left(2\pi x \cdot \frac{1}{2\sqrt{1+y'^2}} \cdot 2y'\right) = 0$$

$$\frac{d}{dx}\left(\frac{xy'}{\sqrt{1+y'^2}}\right) = 0 \qquad \frac{0}{2\pi} = 0$$

$$\frac{xy'}{\sqrt{1+y'^2}} = c_1$$

$$\therefore x^2 y'^2 = c_1^2 (1+y'^2)$$

$$\therefore \frac{x^2 y'^2}{1+y'^2} = c_1^2$$

$$\frac{x^2}{c_1^2} = \frac{1+y'^2}{y'^2} = \frac{1}{y'^2}+1$$

$$\frac{x^2}{c_1^2} - 1 = \frac{1}{y'^2}$$

$$y'^2 = \frac{c_1^2}{x^2-c_1^2}$$

$$y' = \frac{c_1}{\sqrt{x^2-c_1^2}}$$

$$y' = \frac{1 \cdot c_1}{\sqrt{\frac{x^2}{c_1^2}-1}}$$

on inte.

$$y = \cosh^{-1}\left(\frac{x}{c_1}\right) + c_2$$

$$\therefore x = c_1 \cosh(y - c_1)$$

which is eq^n of catenary.

5] Find the extremal of the functional $I(y) = \int_0^{\pi/2} (y'^2 - y^2 + 2xy)\, dx$ subject to the cond^n

$y(0) = 0$ & $y(\pi/2) = 0$.

→ Let,

Given, $I(y) = \int_0^{\pi/2} (y'^2 - y^2 + 2xy)\, dx$

Here, $f(x, y, y') = y'^2 - y^2 + 2xy$

f gives extremal value if $\Rightarrow$ If f satisfies Euler's Langrange's eq^n,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$(-2y + 2x) - \frac{d}{dx}(2y') = 0$$

$$-2y + 2x - 2y'' = 0$$

Dividing by 2.0

$$y'' + y = x$$

$$\frac{d^2y}{dx^2} + y = x$$

$$(D^2+1)\,y = x$$

Auxiallary eq^n is,

$$D^2 + 1 = 0$$

$$m^2 + 1 = 0$$

$$m^2 = -1$$

$$m = \pm i$$

$\therefore$ C.F. $= c_1 \cos x + c_2 \sin x$

P.I. $= \frac{1}{D^2+1} x = (1 + D^2)^{-1} x$

$= [1 - D^2 + D^4 - D^6 + \dots]\, x$

$= [x - 0]$

P.I. $= x$

$=$

$\therefore y =$ C.F. $+$ P.I

$y = c_1 \cos x + c_2 \sin x + x$

Given,

$y(0) \Rightarrow c_1 \cos(0) + c_2 \sin(0) + 0$

$= c_1 + 0$

$\Rightarrow c_1 = 0$

$y(\pi/2) = c_1 \cos(\pi/2) + c_2 \sin(\pi/2) + \pi/2$

$= c_1(0) + c_2(1) + \pi/2$

$y(\pi/2) = c_2 + \pi/2 \qquad \therefore c_2 = -\pi/2$

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$$\therefore\ y = -\frac{\pi}{2}\sin x + x$$

6) Find the extremal of the functional $I(y) = \int_1^2 \frac{x^3}{(y')^2}\,dx$ subject to $y(1)=0,\ y(2)=3$

→ Let,

Given, $I(y) = \int_1^2 \frac{x^3}{(y')^2}\,dx$

$\frac{1}{(y')^2} = (y')^{-2}$

$-2y'$

Here, $f(x, y, y') = x^3/y'^2$

If f satisfies EL eqn,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$0 - \frac{d}{dx}\left(x^3 \cdot (-2y')\right) = 0$$

$$\frac{d}{dx}\, x^3 \cdot 2y' = 0$$

$2/d^2/dx^2\, x^3 y' = 0 \Rightarrow x^3 \cdot 2y' = c_1$

$d^2/dx^2/x^3 y' = 0 \quad x^3/c_1 = 1/2y'$

$$\Rightarrow y' = \frac{c_1}{2x^3}$$

on integrating,

$$y = \frac{c_1}{2}\cdot\left(\frac{-1}{x}\right) + c_2$$

$$\therefore\ x = -\frac{c_1}{2y} + c_2$$

The Branchistrochrone Problem:-

Statement:-

The Branchistrochrone is curve joining two points in a plane (cannot lying on vertical line) such that a particle falling from the rest under influence of gravity from higher point to lower point in minimum time. The curve is cycloid.

To cover the distance is minimum time then it is →

Find the curve of quickest decent OR State and prove Branchistochrone Problem.

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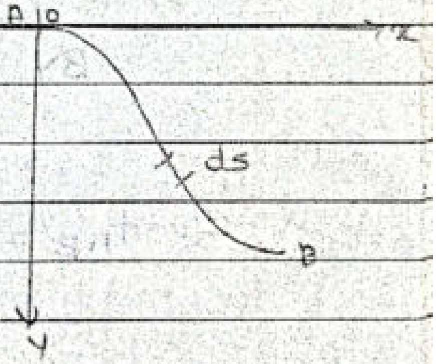
Proof:-

Let A & B be two points not lying in a vertical line.

Let $v = \frac{ds}{dt}$ be the speed of the particle along the curve joining A & B.



$ds = v\,dt$ is the length element on this curve.

We have,

$$dt = \frac{ds}{v}$$

$$= \frac{\sqrt{dx^2+dy^2}}{v}$$

$$\therefore dt = \frac{\sqrt{1+(y')^2}}{v}\,dx$$

The total time required to form a particle from A to B

$$t_{AB} = \int_A^B dt = \int_{x_1}^{x_2} \frac{\sqrt{1+(y')^2}}{v}\,dx \quad \text{---} (1)$$

Total energy is constant at every / any point.

height $h = y$

In this case K.E. & P.E. are given by,
$$T = \frac{1}{2}mv^2 \quad \& \quad V = -mgh\,y$$

By principle of convergence of energy,

from conservation thm $T+V = \text{constant}$

$\therefore T + V = \text{constant}$

At initial position $v = 0$ (i.e., particle is at rest) & $y = 0$

$$\therefore T = 0 \quad , \quad V = 0$$

$$\therefore T + V = \text{constant} = 0$$

$$\therefore T = -V$$

$$\frac{1}{2}mv^2 = -(-mgy)$$

$$v^2 = \frac{2mgy}{m}$$

$$\therefore v = \sqrt{2gy}$$

By eqn ① $t_{AB} = \int_{x_1}^{x_2} \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}\, dx$

$dx^2 + dy^2$

$\left[1+\left(\frac{dy}{dx}\right)^2\right] dx$

$\left(\frac{dx}{dy}+1\right) dx$

$$\therefore t_{AB} = \int_{y_1}^{y_2} \frac{\sqrt{1+(x')^2}}{\sqrt{2gy}}\, dy$$

$$= \frac{1}{\sqrt{2g}} \int_{y_1}^{y_2} \frac{\sqrt{1+(x')^2}}{\sqrt{y}}\, dy$$

$$= \frac{1}{\sqrt{2g}} \int_{y_1}^{y_2} f(y, x, x')\, dy \quad \text{——②}$$

where,

$$f = \sqrt{\frac{1+(x')^2}{y}}$$

this funn integral is minimal if f satisfies Euler Lagrange's eqn

$$\frac{\partial f}{\partial x} - \frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right) = 0$$

$$0 - \frac{d}{dy}\left(\frac{1}{\sqrt{y} \cdot 2\sqrt{1+(x')^2}} \cdot 2x'\right) = 0$$

$$\frac{x'}{\sqrt{y} \cdot \sqrt{1+(x')^2}} = \text{constant} = c_1$$

$$\frac{(x')^2}{(1+x'^2) \cdot y} = c_1^2$$

$$\frac{1}{\left(\frac{1}{x'^2}+1\right)} = c_1^2 y$$

$$\frac{1}{x'^2} + 1 = \frac{1}{c_1^2 y}$$

$$\frac{1}{x'^2} = \frac{1}{c_1^2 y} - 1$$

$$\frac{1}{x'^2} = \frac{1 - c_1^2 y}{c_1^2 y}$$

$$\therefore \quad x' = \sqrt{\frac{c_1^2 y}{1 - c_1^2 y}}$$

$$= \sqrt{\frac{y}{\frac{1}{c_1^2} - y}}$$ Dividing by c_1^2

$$= \sqrt{\frac{y}{a-y}}$$ where $a = \frac{1}{c_1^2}$

$$\therefore \quad x' = \sqrt{\frac{y}{a-y}}$$ $\Rightarrow x' = \frac{dx}{dy}$

$$dx = \sqrt{\frac{y}{a-y}}\, dy \qquad \text{—— (3)}$$

Put $y = a \sin^2 \theta/2$

$$dy = 2a \cos \theta/2 \cdot \frac{1}{2} \cdot d\theta$$

$\therefore\ dy = a \sin\theta/_2 \cdot \cos\theta/_2 \, d\theta$

$\therefore$ By ③

$$dx = \sqrt{\frac{a \sin^2\theta/_2}{a - a\sin^2\theta/_2}} \cdot a \sin\frac{\theta}{2} \cdot \cos\frac{\theta}{2} \, d\theta$$

$$= \frac{\sqrt{a} \ \sin^2\theta/_2}{\sqrt{a}\sqrt{(1-\sin^2\theta/_2)}} \cdot a \sin\theta/_2 \cdot \cos\theta/_2 \, d\theta$$

$$= \frac{\sqrt{a} \ \sin^2\theta/_2}{\sqrt{a} \cdot \cos^2\theta/_2} \cdot a \sin\theta/_2 \cdot \cos\theta/_2 \, d\theta$$

$$dx = a \sin^2\theta/_2 \, d\theta$$

$$dx = a\left(\frac{1-\cos\theta}{2}\right) d\theta$$

On integrating,

$\Rightarrow\ x = \frac{a}{2}\left[\theta - (\sin\theta)\right] + b$ where b is integration constant

$\Rightarrow\ x - b = \frac{a}{2}\left[\theta - \sin\theta\right]$

Now, $y = a\sin^2\theta/_2 = \frac{a}{2}(1-\cos\theta)$

at initial position, $x=0$

$\quad \& \ y=0$

$1-\cos\theta=0$

$\cos\theta=1$

$\therefore\ \theta=0$

$\cos\theta=1$

$\theta=0$

$\& \ 0-b = \frac{a}{2}(\theta-\sin\theta)$

$\cos(0)=1$

$-b = a/_2 (0 - \sin 0)$

$-b = 0$

$\therefore\ b = 0$

$\therefore\ x = \frac{a}{2}(\theta - \sin\theta) \ \& \ y = \frac{a}{2}(1-\cos\theta)$

These are the required parametric eq^ns. of stationary curve. & it is called <u>cycloid</u>.

Show that the geodesic on the surface of right circular cylinder is helix.

→ Let,

Suppose that the cylinder is defined by,

$$x^2+y^2=a^2$$
$$z=z$$

where a is constant.

The parametric eq^n are given by,

$$x=a\cos\theta \quad , \quad y=a\sin\theta \quad , \quad z=z$$

The distance element is given by,

$dx = a\cos\theta$
$dy = -a\sin\theta$

$$ds=\sqrt{dx^2+dy^2+dz^2}$$
$$=\sqrt{a^2d\theta^2+dz^2}$$

∴ Total length of the curve joining the points $P\theta_1 z_1$ & $Q\theta_2 z_2$ is given by,

$$s=\int_{P}^{Q} ds$$

$$=\int_{\theta_1}^{\theta_2}\sqrt{a^2+z'^2}\,d\theta$$

$\sqrt{a^2d\theta^2+dz^2}$
$=\sqrt{a^2+\left(\frac{dz^2}{d\theta^2}\right)^2}$
$=\sqrt{a^2+(z')^2}$

$$=\int_{\theta_1}^{\theta_2} f(\theta, z, z')\,d\theta$$

where, $f=\sqrt{a^2+z'^2}$

The curve in geodesic if f satisfies Euler's Langrange's eq^n.

$$\frac{\partial f}{\partial z}-\frac{d}{d\theta}\left(\frac{\partial f}{\partial z'}\right)=0$$

$$0-\frac{d}{d\theta}\left(\frac{1}{2\sqrt{a^2+(z')^2}}\cdot 2z'\right)=0$$

$$\frac{z'}{\sqrt{a^2+(z')^2}}=\text{constant}=c$$

In 3 dimensional

$x = a\sin\theta\cdot\cos\phi$
$y = a\sin\theta\cdot\sin\phi$
$z = a\cos\theta$

$$\therefore (z')^2 = c^2\cdot(a^2 + z'^2)$$
$$\therefore (z')^2(1-c^2) = c^2a^2$$
$$\therefore (z')^2 = \frac{c^2a^2}{1-c^2}$$
$$z' = \frac{ca}{\sqrt{1-c^2}} = c_1$$
$$\therefore z' = c_1$$

$z' = \frac{dz}{d\theta}$

$$\therefore dz = c_1\, d\theta$$

On integrating,

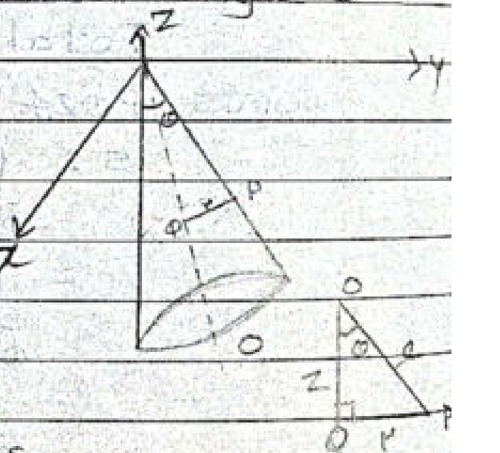
$$z = c_1\theta + c_2$$

which is required eq^n of helix.

3) Find the diff. eq^n. of the geodesic on the surface of inverted cone with semi-vertical angle θ.

→ Let,

Any point (x, y, z) on surface of cone will satisfy,



$\sin\theta = \frac{r}{l}$

$r = l\sin\theta$

$\cos\theta = \frac{z}{l}$

$l = z\cos\theta$

$\tan\theta = \frac{r}{z}$

$r = z\tan\theta$

$$x^2 + y^2 = z^2\tan^2\theta$$

The parametric eq^n are,

$x = ar\cos\phi$
$y = ar\sin\phi$
$z = br$

θ is constants.

where a & b are constants such that $a = \sin\theta$ & $b = \cos\theta$

So, r & ϕ are generalized co-ordinates

If we consider a curve on cone then the distance element on this curve is given by,

$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$
$$= \sqrt{dr^2 + a^2r^2d\phi^2}$$

θ is constant $\therefore \dot{\theta} = 0$

$dx =$

$dx^2 + dy^2 + dz^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$
$= \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2$

FaceLook

$\therefore$ The total length of the curve joining point $A(r_1, \phi_1)$ & $B(r_2, \phi_2)$ on the cone is,

$$S = \int_A^B ds$$

$$= \int_{\phi_1}^{\phi_2} \int_{r_1}^{r_2} \sqrt{1+a^2r^2\phi'^2} \cdot dr$$

Taking dr common out $\frac{\sqrt{dr^2+a^2r^2 d\phi^2}}{dr} = \frac{\sqrt{r\ 1+a^2r^2 d\phi'}}{dr}$

$$= \int_{r_1}^{r_2} f(r, \phi, \phi')\, dr$$

where $f = \sqrt{1+a^2r^2\phi'^2}$

The curve is geodesic if f satisfies Euler's Langrange's eq^n,

$$\frac{\partial f}{\partial \phi} - \frac{d}{dr}\left(\frac{\partial f}{\partial \phi'}\right) = 0$$

$$0 - \frac{d}{dr}\left(\frac{2\phi' a^2 r^2}{2\sqrt{1+a^2r^2\phi'^2}}\right) = 0$$

$$\frac{d}{dr}\left(\frac{a^2r^2\phi'}{\sqrt{1+a^2r^2(\phi')^2}}\right) = c$$

$$\Rightarrow \frac{a^2r^2\phi'}{\sqrt{1+a^2r^2(\phi')^2}} = C$$

$$\frac{a^4r^4\phi'^2}{1+a^2r^2\phi'^2} = C^2$$ → squaring on Num & deno.

$$\frac{1}{\frac{1}{a^4r^4\phi'^2} + \frac{1}{a^2r^2}} = C^2$$

$$\frac{1}{a^4r^4(\phi')^2} + \frac{1}{a^2r^2} = \frac{1}{C^2}$$

$$\Rightarrow \frac{1}{a^4r^4(\phi')^2} = \frac{1}{C^2} - \frac{1}{a^2r^2}$$

$$\frac{1}{a^4 r^4 \phi'^2} = \frac{a^2 r^2 - c^2}{a^2 r^2 c^2}$$

$$\therefore \quad (\phi')^2 = \frac{c^2}{a^2 r^2 (a^2 r^2 - c^2)}$$

$$\therefore \quad \phi' = \frac{c}{ar(\sqrt{a^2 r^2 - c^2})}$$

$$\phi' = \frac{c}{a^2 r \sqrt{r^2 - c^2/a^2}}$$

$\sqrt{a^2 r^2 - c^2}$
$= \sqrt{a^2 (r^2 - c^2/a^2)}$
$= a\sqrt{}$

$$\phi = \frac{c}{a^2} \cdot \int \frac{1}{r\sqrt{r^2 - c^2/a^2}}$$

$$= \frac{c}{a^2} \cdot \int \frac{1}{r\sqrt{r^2 - c_1^2}}$$

$\int \frac{1}{x\sqrt{x^2 - a^2}}$
$= \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right)$

$$= \frac{c}{a^2 (c/a)} \sec^{-1}\left(\frac{r}{c/a}\right) + c_1$$

$$\phi = \frac{1}{a} \sec^{-1}\left(\frac{ar}{c}\right) + c_1$$

$$a\phi = \sec^{-1}\left(\frac{ar}{c}\right) + ac_1$$

$$a\phi - ac_1 = \sec^{-1}\left(\frac{ar}{c}\right)$$

$$\frac{ar}{c} = \sec(a\phi - ac_1)$$

$$\therefore \quad r = \frac{c}{a} \sec(a\phi - ac_1)$$

which is required eq^n of geodesic

4) Find the extremal of $I(y) = \int^{\log 2} (e^{-x} y'^2 - e^{x} \cdot y^2)\, dx$.

(hint - substitute $x = \log z$ in $y'' - y' + e^{2x} y = 0$)

→ Let,

Here, $f(x, y, y') = e^{-x} y'^2 - e^{x} y^2$

integral I is extremal if f satisfies Euler's langrage's eqⁿ,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$-2e^{x} \cdot y - \frac{d}{dx}\left(2 e^{-x} y'\right) = 0$$

$$-2e^{x} \cdot y - (-2e^{-x} y' + 2e^{-x} y'') = 0$$

$$-2e^{x} y + 2e^{-x} y' - 2e^{-x} y'' = 0$$

$\cancel{-2e^{2x}}$ ($\cancel{-y}$ - dividing/multiplying by $\frac{e^{x}}{2}$

$$-e^{2x} \cdot y + y' - y'' = 0$$

$$y'' - y' + e^{2x} \cdot y = 0 \quad \text{—— ①}$$

Put $x = \log z$

$z = e^{x}$

$\therefore \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$ → By using change of parameter property

$\therefore \frac{dy}{dx} = \frac{dy}{dz} \cdot e^{x}$ $\quad \frac{d}{dx}\left(e^{x} \cdot \frac{dy}{dz}\right)$

$$\frac{d^2y}{dx^2} = e^{x} \frac{dy}{dz} + \frac{d}{dx}\left(\frac{dy}{dz}\right) e^{x}$$

$$\frac{d^2y}{dx^2} = e^{x} \frac{dy}{dz} + e^{x} \cdot \frac{d}{dz}\left(\frac{dy}{dx}\right) \cdot \frac{dz}{dx}$$

$$= e^{x} \cdot \frac{dy}{dz} + e^{x} \cdot \frac{d^2y}{dz^2} \cdot e^{x}$$

$$\frac{d^2y}{dx^2} = e^{x} \cdot \frac{dy}{dz} + e^{2x} \frac{d^2y}{dz^2} \quad \text{—— ①}$$

Eqⁿ ① becomes,

$$e^{2x}\cdot\frac{d^2y}{dz^2} + e^{x}\cdot\frac{dy}{dz} - e^{x}\cdot\frac{dy}{dz} + e^{2x}\cdot y = 0$$

$$e^{2x}\frac{d^2y}{dz^2} + e^{2x}\cdot y = 0$$

$$\frac{d^2y}{dz^2} + y = 0$$

$$(D^2+1) = 0 \Rightarrow$$

$$D^2 = \pm i$$

$$C.F. = c_1\cos z + c_2\sin z$$

$$P.I. = \frac{1}{(D^2+1)}(0) = 0$$

$$\therefore y = c_1\cos z + c_2\sin z$$

$$\therefore y = c_1\cos e^{x} + c_2\sin e^{x}$$

5) Show that the time taken by a particle moving along a curve $y = y(x)$ with velocity $\frac{ds}{dt} = 5x$ from point $(1, 1)$ to $(0, 0)$ is minimum if the curve is circle having it's centre on x-axis

$dt = \frac{ds}{x}$

6) Show that Euler's Langrange's Eq^n of the function $I(y) = \int_{x_1}^{x_2} f(x, y, y')\,dx$ has the 1^{st} integral, $f - y'\dfrac{\partial f}{\partial y'}$ = constant if the integrand does not depends explicitely on x.

→ Let,

Given, $\dfrac{\partial f}{\partial x} = 0$ → bcoz it is does not depends on x. ——①

Now EL equations are,

$$\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Now consider,

$$\frac{d}{dx}\left(f - y'\frac{df}{\partial y'}\right) = 0$$

$$= \frac{df}{dx} - y''\frac{\partial f}{\partial y'} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\cdot y'$$

$y' = \dfrac{\partial y}{\partial x}$

$\dfrac{\partial y'}{\partial x} = y''$

$$= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' - y''\frac{\partial f}{\partial y'} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

x is fun^n of x, y, y' ∴ its der is $\dfrac{\partial f}{\partial x}\cdot\dfrac{\partial x}{\partial x} + \dfrac{\partial f}{\partial y}\cdot\dfrac{\partial y}{\partial x} + \dfrac{\partial f}{\partial y'}\cdot\dfrac{\partial y'}{\partial x}$

$$= \frac{\partial f}{\partial x}\cdot 1 + \frac{\partial f}{\partial y}y' - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\cdot y'$$

$$= y'\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]$$

$$= y'(0)$$

$$= 0$$

$$\therefore \frac{d}{dx}\left[f - y'\left(\frac{\partial f}{\partial y'}\right)\right] = 0$$

$$\therefore f - y'\left(\frac{\partial f}{\partial y'}\right) = \text{constant}$$

is the first integral of given EL eq^n.

1) Find the extremal of the functional

$$I(y) = \int_{x_0}^{x_1} \frac{1+y^2}{y'^2}\, dx,$$

$\frac{1}{(y')^2}$

$\frac{1}{x^2} = x^{-2}$

$-2y'^{-1}$

$\frac{-2}{y'}$

→ Let,

Here, $\frac{\partial f}{\partial x} = 0$

∴ We have, $f - y'\frac{\partial f}{\partial y'} = C_1$ as a 1^{st} integral.

$$\frac{1+y^2}{y'^2} - y'(1+y^2)\cdot\frac{(-2)}{y'^3} = C_1$$

$$+\left[\frac{1+y^2}{y'^2} + \frac{2(1+y^2)}{y'^2}\right] = C_1$$

$$\frac{1+y^2}{y'^2}(1+2) = C_1$$

$$\frac{1+y^2}{y'^2}(3) = C_1$$

$$y' = \frac{3}{C_1}\sqrt{1+y^2}$$

$$\therefore y' = C_2\sqrt{1+y^2}$$

$\frac{3}{C_1} = C_2$ be any constant

$$\therefore \frac{dy}{dx} = C_2\sqrt{1+y^2}$$

$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$\frac{dy}{\sqrt{1+y^2}} = C_2\, dx$$

on inte.

$$\sinh^{-1} y = C_2 x + C_3$$

$$\therefore y = \sinh(C_2 x + C_3)$$

$\int \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$ | $\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$ $\sinh = \frac{1}{\sqrt{a^2+x^2}}$ $2\int\frac{y}{\sqrt{1+y^2}} = 2\sqrt{y}$ $\sqrt{x} = 2\sqrt{x}$

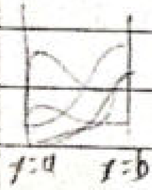
*8) A case of variable end points along a vertical line $x=a$ & $x=b$.

Among all the curves whose end points lies on two given vertical lines $x=a$ & $x=b$, Find the curve for which the functional $I(y) = \int_a^b f(x, y, y')\,dx$ has an extremum.

→ Let,

Suppose that the curve $y = y(x)$ whose end points lie on $x=a$ & $x=b$ extremizes

$$I(y) = \int_a^b f(x, y, y')\,dx \quad \text{——— ①}$$



The neighbouring curves may be defined by the family.

$y'(x) = 0 + \alpha(\eta(x))$

$$y(x, \alpha) = y(x) + \alpha \cdot \eta(x) \quad \text{at } \alpha = 0 \quad \text{——— ②}$$

where α is parameter and $\eta(x)$ is arb fun^n (note that $\eta(a) \neq 0$ & $\eta(b) \neq 0$).

The value of integral ① along family ② is,

$$I(\alpha) = \int_a^b f(x, y(x,\alpha), y'(x,\alpha))\,dx \quad \text{——— ③}$$

This integral is extremum if $\alpha = 0$

$f(x)$ is extremum iff $f'(x) = 0$

$$I'(\alpha)\Big|_{\alpha=0} = 0$$

$$0 = \int_a^b \left[\frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'} \cdot \frac{\partial y'}{\partial \alpha}\right] dx$$

$$= \int_a^b \left[\frac{\partial f}{\partial y} \cdot \eta(x) + \frac{\partial f}{\partial y'} \cdot \eta'(x)\right] dx$$

$$[f(x,t)]_{t=b(x)} \frac{db}{dx} + f[x,t] \cdot \frac{da}{dx}$$

By DUIS Rule we can write,

$$= \int_a^b \frac{\partial f}{\partial y} \eta(x)\, dx + \left[\frac{\partial f}{\partial y'} \eta(x)\right]_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \cdot \eta(x) dx$$

$$= \int_a^b \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'}\right] \eta(x)\, dx$$

$$+ \left[\frac{\partial f}{\partial y'} (\eta(x))\right]_a^b \quad \text{—④}$$

Note that the term $\left[\eta(x) \frac{\partial f}{\partial y'}\right]$ is not zero

as $\eta = a \neq 0$ & $\eta = b \neq 0$.

$\therefore$ If we put the cond^n as,

$$\left.\frac{\partial f}{\partial y'}\right|_{x=a} = 0 \quad \& \quad \left.\frac{\partial f}{\partial y'}\right|_{x=b} = 0 \quad \text{—⑤}$$

$\therefore$ By ④

$$\int_a^b \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'}\right) \eta(x) dx = 0$$

By basic Lemma, $\int_a^b g(x) \cdot \eta(x)\, dx \Rightarrow g(x) = 0$

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0 \quad \text{—⑥}$$

Thus to solve the variable end point problem we have to solve eq^n ⑥ along with the cond^n ⑤.

Note that,

If the one end point say $x = a$ is fixed and other end point $x = b$ is variable then we say that we have to solve Euler Langrange's eq^n ⑥ along with the cond^n

$$\left.\frac{\partial f}{\partial y'}\right|_{x=b} = 0$$

FaceLook

Thm:-

Consider the functional $I(y) = \int_a^b f(x, y, y', y'')\,dx$

the curve y extremizes this integral if the Euler Langrange's eq^n $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) =$

is satisfied and

1] $\frac{\partial f}{\partial y''} = 0$ at the end points if y' is not prescribed there.

2] $\frac{\partial f}{\partial y'} - \frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right) = 0$ at the end points if y is not prescribed, and y' is prescribed at end points.

→ Let,

$P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two fixed points in XY plane. The points P & Q can be join by infinitely many curves. According to this values of the integral I will be diff^n for diff^n path.

We shall look for a curve along which the functional I has an extremum value.

Let 'C' be the curve bet^n P & Q whose eq^n is given by,

$y = y(x)$ which gives extremum value of functional $I(y(x)) = \int_{x_1}^{x_2} f(x, y, y', y'')\,dx$ —①

Now the neighbouring family is defined by

$y(x, \alpha) = y(x) + \alpha\,\eta(x)$ $\forall\ \alpha$ is parameter

& $\eta(x) = C^2$ fun^n

$\eta(x)$: C^2 fun^n bcz we get double deri.

For diff^n values of α we get diff^n curves.

Part I] :

Since y is prescribed at the end points and y' is not prescribed at the end points.

at the end points

$\Rightarrow$ $y(x_1, \alpha) = y(x_1)$ & $y(x_2, \alpha) = y(x_2)$

$\Rightarrow$ $\eta(x_1) = 0$ & $\eta(x_2) = 0$

not prescribed $\eta'(x_1) \neq 0$ $\eta'(x_2) \neq 0$

But, $\eta'(x_1) \neq 0$ —③ & $\eta'(x_2) \neq 0$ —④

The value of the functional along the neighbouring curves

$$I(y(x, \alpha)) = \int_{x_1}^{x_2} f(x, y(x, \alpha), y'(x, \alpha), y''(x, \alpha))\, dx$$

The value of the functional I is extremum if,

$$\left(\frac{\partial I}{\partial \alpha}\right)\Big|_{\alpha=0}$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \alpha} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\cdot\frac{\partial y'}{\partial \alpha} + \frac{\partial f}{\partial y''}\cdot\frac{\partial y''}{\partial \alpha}\right) dx = 0$$

$$0 = \int_{x_1}^{x_2} \left[0 + \eta(x)\frac{\partial f}{\partial y} + \frac{\partial f}{\partial y'}\cdot \eta'(x) + \frac{\partial f}{\partial y''}\,\eta''(x)\right] dx$$

$$0 = \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\,\eta(x)\, dx + \left[\frac{\partial f}{\partial y'}\cdot\eta(x)\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\cdot\eta(x)\, dx$$

$$+ \left[\frac{\partial f}{\partial y''}\cdot\eta'(x)\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right)\cdot\eta'(x)\, dx$$

$$\left[\frac{\partial f}{\partial y''}\cdot\eta(x)\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{d}{dx}\frac{\partial f}{\partial y''}\right)\cdot\eta(x)\, dx$$

$$\int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta(x)\,dx+\left[\frac{\partial f}{\partial y'}\cdot\eta(x)\right]_{x_1}^{x_2}+\left[\frac{\partial f}{\partial y''}\,\eta'(x)\right]_{x_1}^{x_2}$$

$$-\left[\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\cdot\eta(x)\right]_{x_1}^{x_2}+\int_{x_1}^{x_2}\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)\cdot\eta(x)\,dx$$

$$=0$$

$$\int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)\right]\eta(x)\,dx$$

$$+\left[\left[\frac{\partial f}{\partial y'}-\frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right)\right]\eta(x)\right]_{x_1}^{x_2}+\left[\frac{\partial f}{\partial y''}\cdot\eta'(x)\right]_{x_1}^{x_2}$$

$$=0 \qquad \text{--- (5)}$$

Since y' is not prescribed at the end point i.e. $\eta'(x_1)\neq 0$, & $\eta'(x_2)\neq 0$.

If we put the condition that $\left[\frac{\partial}{\partial y''}\right]\Big|_{x=x_1}=0$ & $\left[\frac{\partial}{\partial y''}\right]\Big|_{x=x_2}=0$ --- (6)

from eq^n (5), (6) & (3)

$$\int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)\right]\cdot\eta(x)\,dx=0$$

$\int_a^b g(x)\cdot\eta(x)\,dx$ $\ni g(x)=0$

By basic lemma of calculation of variation,

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)=0 \qquad \text{--- (7)}$$

$\therefore$ The functional I extremizes if f satisfies EL eq^n (7) along with the cond^n (6) when y' is not prescribed at the end points.

Part II]

Eq^n (5) is,

$$\int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)\right]\eta(x)\,dx$$

$$+\left\{\frac{\partial f}{\partial y'}-\frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right)\right]\eta(x)\Big\}_{x_1}^{x_2}+\left[\frac{\partial f}{\partial y''}\cdot\eta'(x)\right]_{x_1}^{x_2}$$

$$=0$$

Since y is not prescribed at the end points we have, $\eta(x_1)\neq 0$ & $\eta(x_2)\neq 0$.
and as y' is prescribed at the end points we have,

$$\eta'(x_1)=0 \quad \& \quad \eta'(x_2)=0. \qquad \text{—— (8)}$$

If we put $\left[\frac{\partial f}{\partial y'}-\frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right)\right]_{x_1}^{x_2}=0$ —— (9)

$\therefore$ By using eq^n (9), (8) in eq^n (5),

$$\int_{x_1}^{x_2}\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)\eta(x)\,dx=0$$

By basic lemma of calculus of variation,

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)=0 \qquad \text{—— (10)}$$

$\therefore$ The functional I is extremum if f satisfies EL eq^n (10) along with the cond^n (9) when y is not prescribed at the end points.

Generalization :-

The 2n-times differentiable fun^n y extremizes the integral,

$$I(y) = \int_{x_1}^{x_2} f(x, y, y' \dots y^{(n)})\, dx$$

where $y^{(j)}$ are prescribed at the end points $j = 0, 1, 2, \dots n$. If the following Euler's Langrange's eq^n,

$$\frac{\partial f}{\partial y} - \underset{(-1)^1=-1}{\frac{d}{dx}}\left(\frac{\partial f}{\partial y'}\right) + \underset{(-1)^2=+1}{\frac{d^2}{dx^2}}\left(\frac{\partial f}{\partial y''}\right) - \underset{(-1)^3=-1}{\frac{d^3}{dx^3}}\left(\frac{\partial f}{\partial y'''}\right)$$

$$+ \dots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial f}{\partial y^n}\right).$$

Q] Extremize $I(y) = \int^1 y''^2 - 2xy\, dx$ subject to $y(0) = y'(0) = 0$ & $y(1) = \frac{1}{120}$ & $y'(1)$ is not prescribed.

→ Let,

Given, $I(y) = \int_0^1 y''^2 - 2xy\, dx$

Here, $f(x, y, y', y'') = y''^2 - 2xy$

∴ Integral I is extremum if f satisfies EL eq^n.

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0$$

$$-2x - \frac{d}{dx}(0) + \frac{d^2}{dx^2}(2y'') = 0$$

$$-2x + \frac{d^2}{dx^2} 2y'' = 0$$

$$2\left(\frac{d^2}{dx^2} y'' - x\right) = 0$$

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$$\frac{d^2}{dx^2} y'' + x = 0$$

$$\frac{d^4}{dx^4}(y) \cancel{+} = x$$

Auxillary eq^n is,

$$D^4 y = x$$

$$\therefore \quad m^4 = 0 \quad , m = 0, 0, 0, 0$$

$$C.F. = (c_1 \cos(0) + c_2 x + c_3 x^2 + c_4 x^3) e^{mx}$$

$$= (c_1 + c_2 x + c_3 x^2 + c_4 x^3) e^{0x}$$

$$P.I. = \frac{1}{D^4}(x) = \frac{1}{D}\left(\frac{x^2}{2}\right)$$

$$= \frac{x^4}{4} \cdot \frac{1}{D^3}\left(\frac{x^2}{2} + c_5\right) +$$

$$= \frac{1}{D^2}\left(\frac{x^3}{6}\right)$$

$$= \frac{1}{D} \frac{x^4}{24}$$

$$P.I. = \frac{x^5}{120}$$

$$\therefore y = C.F. + P.I.$$

$$\therefore y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + \frac{x^5}{120}$$

$$y(0) = c_1$$

$$0 = c_1 \quad \Rightarrow \quad c_1 = 0$$

$$y'(0) = c_2 + 2x c_3 + 3x^2 c_4 + \frac{1}{120} 5x^4$$

$$0 = c_2$$

$$\Rightarrow c_2 = 0$$

$y(1) = \frac{1}{120} \Rightarrow c_1 + c_2 + c_3 + c_4 + \frac{1}{120} = \frac{1}{120}$

$0 + 0 + c_3 + c_4 = \frac{1}{120} - \frac{1}{120}$

$c_3 + c_4 = 0$

$y'(1)$ is not prescribed

Now, $y'(1)$

$\frac{\partial f}{\partial y''}\Big|_{x=1} = 0 \Rightarrow 2y''\Big|_{x=1} = 0 \Rightarrow y''(1) = 0$

$\therefore y' = c_2 + 2c_3 x + 3c_4 x^2 + \frac{x^4}{24}$

$y'' = 2c_3 + 6c_4 x + \frac{4x^3}{24}$

$y''(1) = 2c_3 + 6c_4 + \frac{4}{24} = 0$

$2c_3 + 6c_4 = \frac{-1}{6}$

$2(c_3 + 3c_4)$

$\therefore c_3 + 3c_4 = \frac{-1}{12}$

solving

$c_3 + c_4 = 0$

$-\; c_3 + 3c_4 = -1/12$

$-2c_4 = +1/12$

$c_4 = \frac{1}{12 \times 2}$

$\therefore c_4 = \frac{-1}{24}$

$c_3 = \frac{1}{24}$

$\therefore y = \frac{1}{24}x^2 - \frac{1}{24}x^3 + \frac{x^5}{120}$ is required curve

3) Extremize $I(y) = \int_a^b f(x, y) \cdot \sqrt{1+y'^2}\, dx$.

→ Let,

Given $I(y) = \int_a^b f(x, y) \cdot \sqrt{1+y'^2}\, dx$

We write, $I(y) = \int_a^b g(x, y, y')\, dx$

$$g(x, y, y') = f(x, y) \cdot \sqrt{1+y'^2}$$

$\sqrt{1+y'^2}\ f'$ $\quad \frac{\partial}{\partial y}(f(x,y)) = \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$

∴ I is extremum if g satisfies EL eqⁿ,

$$\therefore \frac{\partial g}{\partial y} - \frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right) = 0$$

$$\sqrt{1+y'^2}\ \frac{\partial f}{\partial y} - \frac{d}{dx}\left(f(x, y) \cdot \frac{1}{2\sqrt{1+y'^2}} \cdot 2y'\right) = 0$$

$$\therefore \sqrt{1+y'^2}\ \frac{\partial f}{\partial y} - \frac{d}{dx}\ \frac{f(x, y) \cdot y'}{\sqrt{1+y'^2}} = 0 \quad \text{--- (1)}$$

Now,

$$\frac{d}{dx}\left(\frac{f(x, y) \cdot y'}{\sqrt{1+y'^2}}\right)$$

$\frac{1}{\partial x} f(x,y)$

$$= f(x, y) \cdot \frac{y'}{\sqrt{1+y'^2}}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\, y'\right) +$$

$\frac{\partial f}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x}$

$$f(x, y)\left[\frac{\sqrt{1+y'^2} \cdot y'' - y' \cdot \frac{1}{2\sqrt{1+y'^2}} \cdot 2y' \cdot y''}{\left(\sqrt{1+y'^2}\right)^2}\right]$$

$$= \frac{y'}{\sqrt{1+y'^2}}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\, y'\right) + f(x, y)\left[\frac{(1+y'^2) \cdot y'' - y'^2 \cdot y''}{(1+y'^2)\sqrt{1+y'^2}}\right]$$

$$= \frac{y'}{\sqrt{1+y'^2}}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\, y'\right) + f(x, y)\left[\frac{y''}{(1+y'^2)\sqrt{1+y'^2}}\right] \quad \text{--- (2)}$$

$\therefore$ From eq^n ①,

$$\sqrt{1+y'^2}\,\frac{\partial f}{\partial y}-\left(\frac{+y'}{\sqrt{1+y'^2}}\right)\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}y'\right)+f(x,y)\left[\frac{y''}{(1+y'^2)\sqrt{1+y'^2}}\right.$$

$$\sqrt{1+y'^2}\,\frac{\partial f}{\partial y}-\frac{y'}{\sqrt{1+y'^2}}\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}y'\right)-f(x,y)\left[\frac{y''}{(1+y'^2)\sqrt{1+y'^2}}\right.$$

Dividing by $\sqrt{1+y'^2}$

$$(1+y')^2\frac{\partial f}{\partial y}-y'\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}y'\right)-f(x,y)\left[\frac{y''}{(1+y'^2)}\right.$$

$$\frac{\partial f}{\partial y}-y'\frac{\partial f}{\partial x}-f(x,y)\cdot\frac{y''}{(1+y'^2)}=0$$

is the required eq^n.

4) Extremize $I(x(t))=\frac{1}{2}\int_0^2(\ddot{x})^2\,dt$ subject to $x(0)=1$ & $\dot{x}(0)=1$ & $x(2)=1$ & $\dot{x}(2)=0$.

$\rightarrow$ Let,

Given $I(x,t)=\frac{1}{2}\int_0^2(\ddot{x})^2\,dt$

Here, $f(t,x,\dot{x},\ddot{x})=\frac{(\ddot{x})^2}{2}$

So, I is extremum if f satisfies EL eq^n

$$\therefore \frac{\partial f}{\partial x}-\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right)+\frac{d^2}{dt^2}\left(\frac{\partial f}{\partial \ddot{x}}\right)=0$$

$$0-\frac{d}{dt}(0)+\frac{d^2}{dt^2}(2\ddot{x})=0$$

$$\left(\frac{d^4}{dt^4}x\right)=0$$

The auxillary eq^n is,

$D^4=0, \quad \Rightarrow m^4=0$

$m = 0, 0, 0, 0.$

$\therefore C.F. = c_1 + c_2 t + c_3 t^2 + c_4 t^3$

$P.I. = \frac{1}{D^4}(0) = 0$

$x = C.F. + P.I.$

$x = c_1 + c_2 t + c_3 t^2 + c_4 t^3$

$x(0) = c_1$

$1 = c_1 \Rightarrow c_1 = 1$

$\dot{x} = c_2 + 2c_3 t + 3c_4 t^2$

$1 = c_2 \Rightarrow c_2 = 1$

$x(2) = c_1 + 2c_2 + 4c_3 + 8c_4$

$1 = c_1 + 2c_2 + 4c_3 + 8c_4$

$1 + 2 + 4c_3 + 8c_4 = 1$

$4c_3 + 8c_4 = -2$

$4(c_3 + 2c_4) = -2$

$c_3 + 2c_4 = -2/4$

$c_3 + 2c_4 = -1/2$

$\dot{x}(2) = c_2 + 4c_3 + 12c_4$

$0 = c_2 + 4c_3 + 12c_4$

$1 + 4c_3 + 12c_4 = 0$

$4c_3 + 12c_4 = -1$

on solving,

$4c_3 + 8c_4 = -2$

$4c_3 + 12c_4 = -1$

$-4c_4 = -1 \Rightarrow \underline{\underline{c_4 = \frac{1}{4}}}$

$\therefore 4c_3 + 2 = -2$

$4c_3 = -4$

$\underline{\underline{c_3 = -1}}$

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$$\therefore x = 1 + t - t^2 + \frac{1}{4} t^3$$

5) Extremize $I(y) = \int_0^{\pi/4} y^2 - y'^2 \, dx$ subject to $y(0) = 0$ and right end point can vary along line $x = \pi/4$.

→ Let,

$$I(y) = \int_0^{\pi/4} y^2 - y'^2 \, dx$$

$$\therefore f(x, y, y') = y^2 - y'^2$$

The functional I is extremum if f satisfie EL eq^n.

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$2y - \frac{d}{dx}(2y') = 0$$

$$y'' + y = 0$$

$$(D^2 + 1) y = 0$$

Auxillary eq^n is,

$$D^2 + 1 = 0$$

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

$$\therefore C.F. = c_1 \cos x + c_2 \sin x$$

$$P.I. = \frac{1}{(D^2+1)}(0) = 0$$

$$y = C.F. + P.I.$$

$$y = c_1 \cos x + c_2 \sin x$$

Now,

$$y(0) = c_1 \cos(0) + c_2 \sin(0)$$

$$0 = c_1 \Rightarrow \therefore c_1 = 0$$

FaceLook

$\therefore\ y = c_2 \sin x$

y' is not prescribed & x is variational

$$\left.\frac{\partial f}{\partial y'}\right|_{x=\pi/4} = 0 \qquad \because [\,y \text{ is not prescribed}]$$

$$-2y'|_{x=\pi/4} = 0$$

$$\therefore\ y'(\pi/4) = 0$$

$$y' = c_2 \cos x$$

$$y'(\pi/4) = c_2 \cos(\pi/4)$$

$$c_2 \cos(\pi/4) = 0$$

$$\Rightarrow\ c_2 = 0$$

$$\therefore\ y = 0$$

i.e. extremal is obtained on line $y=0$

6] Prove that the shortest distance betⁿ 2 points in Euclidean free space ($\mathbb{R}^3$) is given by straight line. $r.w$ $\quad dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$

→ Let,

Consider any two points $A(x, y, z)$ & $B(x_2\ y_2\ z_2)$ in $\mathbb{R}^3$,

Infinitesimal distance on this curve is given by,

$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

The total length of this curve joining A & B is,

$$I = \int_A^B ds$$

$$= \int_A^B \sqrt{dx^2 + dy^2 + dz^2}$$

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$$= \int_A^B \sqrt{1+y'^2+z'^2}\, dx$$

Here, $f(x, y, z, y', z') = \sqrt{1+y'^2+z'^2}$

Functional I will be minimum if f satisfies EL eqn,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad \text{—①}$$

$$\& \ \frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right) = 0 \qquad \text{—②}$$

From ①,

$$0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2+z'^2}}\right) = 0$$

$$\therefore \frac{y'}{\sqrt{1+y'^2+z'^2}} = \text{constanct} = c_1$$

From ②,

$$0 - \frac{d}{dx}\left(\frac{z'}{\sqrt{1+y'^2+z'^2}}\right) = 0$$

$$\Rightarrow \frac{z'}{\sqrt{1+y'^2+z'^2}} = c_2$$

$$\therefore y'^2 = c_1^2(1+y'^2+z'^2)$$

$$\therefore y'^2(1-c_1^2) = c_1^2(1+z_1'^2)$$

$$y'^2 = \frac{c_1^2(1+z_1'^2)}{(1-c_1^2)} \qquad \text{—③}$$

$$z'^2 = \frac{-c_2^2(1+y'^2 z_1'^2)}{(1-c_2^2)}$$

$$z'^2 = c_2^2(1+\cancel{z}y'^2+z'^2)$$

$$z'^2(1-c_2^2) = c_2^2(1+y'^2)$$

$$z'^2 = \frac{+c_2^2}{(1-c_2^2)} (1+y'^2) \quad \text{——} ④$$

From ① ③ & ②, ④

$$y'^2 = \frac{c_1^2}{(1-c_1^2)} \left[1 + \left(\frac{+c_2^2}{1-c_2^2} (1+y'^2) \right) \right]$$

$$= \frac{c_1^2}{(1-c_1^2)} \left[1 + \frac{c_2^2}{1-c_2^2} + \frac{c_2^2 y'^2}{1-c_2^2} \right]$$

multiplying $1-c_2^2$ to 1

$$\frac{c_1^2}{(1-c_1^2)} \left[\frac{1}{1-c_2^2} + \frac{c_2^2 y'^2}{1-c_2^2} \right]$$

$$= \frac{c_1^2}{(1-c_1^2)(1-c_2^2)} + \frac{c_1^2 c_2^2 y'^2}{(1-c_1^2)(1-c_2^2)}$$

$$y'^2 \left[1 - \frac{c_1^2 c_2^2}{(1-c_1^2)(1-c_2^2)} \right] = \frac{c_1^2}{(1-c_1^2)(1-c_2^2)}$$

$$y'^2 = \alpha_1 = \text{constant}$$

$$y' = \alpha_1 = \text{constant}$$

$$y = \alpha_1 x + \alpha_2$$

$y = \alpha$

$y' = \beta_2$

Similarly

$$z = \beta_1 x + \beta_2$$

which gives straight line.

7] Show that the geodesic define in 3 dimensional euclidean space $\mathbb{R}^3$ define by the eq^n $x = x(t)$ $y = y(t)$ & $z = z(t)$, is a straight line.

→ Let

Consider two points $A(x_1, y_1, z_1)$ & $B(x_2, y_2, z_2)$

In $\mathbb{R}^3$, if

If $x = x(t)$, $y = y(t)$, $z = z(t)$ are parametric eq^n of the curve joining A & B then infinitesimal distance on this curve is,

$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

$\therefore$ The total length of this curve is,

$$I = \int_A^B ds = \int_A^B \sqrt{dx^2 + dy^2 + dz^2}$$

$\frac{dx^2}{dt^2}$

Multiply & divide by dt^2

$$= \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \; dt$$

where, $\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$, $\dot{z} = \frac{dz}{dt}$

$$\therefore f(t, x, y, z, \dot{x}, \dot{y}, \dot{z}) = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

The functional I will be extremum if f satisfies EL eq^n,

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) = 0$$

$$0 - \frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}\right) = 0$$

$$\frac{\partial f}{\partial \dot{x}} = \text{constant} = c_1$$

$$\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0$$

$$\frac{\partial f}{\partial \dot{y}} = \text{constant} = c_2$$

$$\frac{\partial f}{\partial z} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{z}}\right) = 0$$

$$\frac{\partial f}{\partial \dot{z}} = \text{constant} = c_3.$$

$$\frac{\partial f}{\partial \dot{x}} = c_1 \Rightarrow \frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} = c_1$$

$$\Rightarrow \dot{x} = f c_1$$

$f(t, x, y, z, \dot{x}, \dot{y}, \dot{z}) = \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}$

$$\frac{\partial f}{\partial \dot{y}} = c_2 \Rightarrow \frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} = c_2$$

$$\Rightarrow \dot{y} = f c_2$$

$$\frac{\partial f}{\partial \dot{z}} = c_3 \Rightarrow \frac{\dot{z}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} = c_3$$

$$\Rightarrow \dot{z} = f c_3$$

$$\Rightarrow f = \frac{\dot{z}}{c_3}$$

$$\therefore \dot{y} = \frac{\dot{z}}{c_3} \cdot c_2$$

on integrating,

$$\therefore y = \underbrace{\frac{c_2}{c_3}}_{a_1 = \text{constant}} z + a_2 \qquad \Rightarrow y = a_1 z + a_2 \quad \text{——(1)}$$

simillarly, $\dot{x} = \frac{c_1}{c_3} \cdot \dot{z}$

on integrating,

$$\therefore x = \frac{c_1}{c_3} z + b_2 \quad \Rightarrow \quad x = b_1 z + b_2 \quad \text{——(2)}$$

integration 1st part

Eqⁿ (1) & (2) are eqⁿ of planes and intersection of this plane is a straight line.

Isoperimetric problems:-

The extremization problem of the form extremize $\int_a^b f(x, y, y')\, dx$ subject to the cond^n that $J = \int_a^b g(x, y, y')\, dx = \alpha$ where α is constant, this problems are called as isoperimetric problems.

Such problems are solved by using method of Langrange's Multipliers.

1) Obtain the d.e. satisfied by y which extremizes the integral $I(y) = \int_{x_1}^{x_2} f(x, y, y')\, dx$ subject to conditions $y(x_1) = y_1$, $y(x_2) = y_2$ and the integral $J = \int_{x_1}^{x_2} g(x, y, y')\, dx = \alpha$.

→ Let,

Consider 2 points $P(x_1, y_1)$ & $Q(x_2, y_2)$ in a plane.

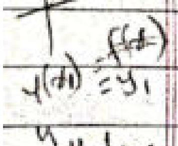
Let, $y(x)$ be the curve passing through P & Q which extremizes $I(y) = \int_{x_1}^{x_2} f(x, y, y')\, dx$ — ①

and satisfying $J(y) = \int_{x_1}^{x_2} g(x, y, y')\, dx = \alpha$. — ②

where α is constant.

Since y passes through P & Q

$y(x_1) = y_1$ & $y(x_2) = y_2$



The neighbouring curves to this y passing through P & Q can be described a C^2 parameter family,

$f(x_1)$

$y(x_1) = y_1$

y value of y at point x_1

$\eta_1(x_1)=0 \quad \eta_1(x_2)=0$
$\eta_2(x_1)=0 \quad \eta_2(x_2)=0$

FaceLook

$$y(x, \epsilon_1, \epsilon_2) = y(x) + \epsilon_1 \eta_1(x) + \epsilon_2 \eta_2(x) \quad \text{--- (3)}$$

$\forall$ ϵ_1 & ϵ_1 are parameters and η_1, η_2 are c' function, satisfying $\eta_j(x_k)=0 \quad \forall j=1,2.$ and $k=1,2.$ --- (4)

The condⁿ (4) ensures that all the curves in family (3) starts at the point P and ends at the points Q.

Note that one parameter family of neighbouring curves need not assures condⁿ (2).

The second parameter ϵ_2 in family (3) can be choosen so that,

$$J(y(x, \epsilon_1 \epsilon_2)) = \alpha.$$

The value of integral (1) along the family (3) is,

$$I(\epsilon_1, \epsilon_2) = \int_{x_1}^{x_2} f(x, y(x, \epsilon_1, \epsilon_2), y'(x, \epsilon_1 \epsilon_2)\, dx$$

and condⁿ (2) becomes,

$$J(\epsilon_1, \epsilon_2) = \int_{x_1}^{x_2} g(x, y(x, \epsilon_1 \epsilon_2), y'(x, \epsilon_1 \epsilon_2)\, dx$$
$$= \alpha \quad \text{--- (6)}$$

We use the method of Langrange's multipliers to extremize (5) under the condⁿ (6)

Define,

$$I^* = I + \lambda J$$

$$\therefore I^* = \int_{x_1}^{x_2} (f + \lambda g)\, dx \quad \text{--- (7)}$$

$$\therefore I^* = \int_{x_1}^{x_2} f^*\, dx$$

Facelook

where λ is called Lagrange's multiplier and $f^* = f + \lambda g$.

According to theory of Lagrange's multiplier extremization of ⑤ under condⁿ ⑥ is equivalent to extremization of ⑦.

I^* is extremum

Now ⑦ is extremum of,

$$\left(\frac{\partial I^*}{\partial \epsilon_j}\right)_{\epsilon_1=\epsilon_2=0} = 0 \qquad j = 1, 2, \ldots$$

f^* is funⁿ of (x, y, y')

i.e.

$$\int_{x_1}^{x_2} \left[\frac{\partial f^*}{\partial y} \cdot \frac{\partial y}{\partial \epsilon_j} + \frac{\partial f^*}{\partial y'} \cdot \frac{\partial y'}{\partial \epsilon_j}\right] dx$$

$\frac{\partial f^*}{\partial x}$ is 0 b/oz x in indpt.

$$\int_{x_2}^{x_2} \left[\frac{\partial f^*}{\partial y} \cdot \eta_j(x) + \frac{\partial f^*}{\partial y'} \cdot \eta_j(x)\right] dx = 0$$

By using integration by parts in 2ⁿᵈ term and using the condⁿ ④ we get

$$\int_{x_1}^{x_2} \frac{\partial f^*}{\partial y} - \frac{d}{dx}\left(\frac{\partial f^*}{\partial y'}\right)\Big] \eta_j(x)\, dx = 0$$

By basic lemma of variational calculus

$\frac{\partial f^*}{\partial y} - \frac{d}{dx}\left(\frac{\partial f^*}{\partial y'}\right) = 0$. are required EL eqⁿ

2) Find the extremals for an isometric problems $I(y) = \int_0^1 y'^2 + x^2\, dx$ subject to condⁿ

$$\int_0^1 y^2\, dx = 2,\ y(0) = 0,\ y(1) = 0.$$

→ Let,

Given $f(x, y, y') = y'^2 + x^2$

$g(x, y, y') = y^2$

To extremize $I(y) = \int_{x_1}^{x_2} f(x, y, y')\, dx$ subject to $J(y) = \int_0^1 g(x, y, y')\, dx = 2$.

Define, $f^* = f + \lambda g$

$$f^* = y'^2 + x^2 + \lambda y^2 \quad \text{--- (1)}$$

where,

λ is Langrange's multiplier.

The EL eq^n is,

$$\frac{\partial f^*}{\partial y} - \frac{d}{dx}\left(\frac{\partial f^*}{\partial y'}\right) = 0$$

$$2\lambda y - \frac{d}{dx}(2y') = 0$$

$$2\left(\lambda y - \frac{d}{dx} y'\right) = 0$$

$$\frac{d}{dx} y' = 0$$

$$y'' - \lambda y = 0$$

$$\therefore (D^2 - \lambda) y = 0$$

Auxillary eq^n is,

$$m^2 - \lambda = 0 \Rightarrow m^2 = \lambda$$

$$\therefore m = \pm\sqrt{\lambda}$$

if $\lambda > 0 \Rightarrow m = \pm\sqrt{\lambda}$

$$\therefore C.F. = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$$

$$P.I. = \frac{1}{D^2 - \lambda}(0) = 0$$

$\therefore y = C.F. + P.I.$

$$\therefore y = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$$

$y(x) =$

If $\lambda = 0 \Rightarrow m = 0, 0$

$$C.F. = c_1 + c_2 x$$
$$P.I. = 0$$
$$\therefore \; y = c_1 + c_2 x$$

if $\lambda < 0 \Rightarrow m = \pm i\sqrt{-\lambda}$

$$\therefore C.F. = c_1 \cos\sqrt{-\lambda}\,x + c_2 \sin\sqrt{-\lambda}\,x$$
$$P.I. = 0$$
$$\therefore \; y = c_1 \cos\sqrt{-\lambda}\,x + c_2 \sin\sqrt{-\lambda}.$$

Now,

$$y = c_1 e^{\sqrt{\lambda}\,x} + c_2 e^{-\sqrt{\lambda}\,x} \quad \text{if } \lambda > 0$$
$$y = c_1 + c_2 x \quad \text{if } \lambda = 0$$
$$y = c_1 \cos\sqrt{-\lambda}\,x + c_2 \sin\sqrt{-\lambda}\,x \quad \text{if } \lambda < 0.$$

Case I:- If $\lambda > 0$ then,

$$y = c_1 e^{\sqrt{\lambda}\,x} + c_2 e^{-\sqrt{\lambda}\,x}$$
$$y(0) = c_1 + c_2$$
$$0 = c_1 + c_2 \Rightarrow c_1 = -c_2$$

$$y(1) = c_1 e^{\sqrt{\lambda}} + c_2 e^{-\sqrt{\lambda}} = 0$$
$$\Rightarrow c_1 (\underbrace{e^{\sqrt{\lambda}} - e^{\sqrt{\lambda}}}_{\neq 0}) = 0$$

i.e. Non ~~zero~~

$e^{x} = \frac{1}{e^{x}} = 0$

$e^{x} = \frac{1}{e^{x}}$

$e^{2x} = 1$

$e^{2x} = 0$

if $x = 0$

$$\Rightarrow c_1 = 0$$
$$\Rightarrow c_2 = 0$$

$y(x) = 0$ which is a trivial sol^n.

In this case,

$\int_0^1 y^2\,dx \neq 2$ so it is not a required sol^n.

Case -II :- If $\lambda = 0$

then $y(x) = c_1 + c_2 x$

$y(0) = c_1 \Rightarrow c_1 = 0$

$y(1) = 1 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 \overset{c_1=0}{=} 0$

$y(x) = 0$. which is again a trivial sol^n but not required sol^n.

Case -III :- If $\lambda < 0$.

The general sol^n of 2 is,

$y = c_1 \cos\sqrt{-\lambda}\, x + c_2 \sin\sqrt{-\lambda}\, x$

$y(0) = c_1$

$0 = c_1 \Rightarrow c_1 = 0$

$y(1) = c_1 \cos\sqrt{-\lambda} + c_2 \sin\sqrt{-\lambda} = 0$

$\Rightarrow c_2 \sin\sqrt{-\lambda} = 0$

$c_2 = 0$ or $\sin\sqrt{-\lambda} = 0$

but If $c_2 = 0$ gives $y(x) = 0$ which is again a trivial sol^n.

$\therefore c_2 \neq 0$,

$\therefore$ We assume that $c_2 \neq 0$

$\therefore \sin\sqrt{-\lambda} = 0$

$\Rightarrow \sqrt{-\lambda} = n\pi$ where $n \in \mathbb{Z}$.

$\therefore \lambda = -n^2\pi^2$

$\therefore$ The general sol^n is,

$y(x) = c_2 \sin(n\pi)x$

Now given that,

$\int_0^1 y^2\,dx = 2 = \int_0^1 c_2^2 \sin^2 n\pi x\,dx$

$= c_2^2 \int_0^1 \sin^2 n\pi x\,dx$

$\sin^2\theta = 2\sin$

$$2 = c_2^2 \int_0^1 \frac{1-\cos 2n\pi x}{2}\, dx$$

$$2 = c_2^2 \int \frac{1}{2}\, dx - \int \cos 2n\pi x\, dx$$

$$2 = c_2^2 \left[\frac{x}{2} - \frac{\sin 2n\pi x}{2n\pi}\right]_0^1$$

$$2 = c_2^2 \left[\frac{1}{2} - \frac{\sin 2n\pi}{2n\pi} - \frac{0}{2} - \frac{\sin(0)}{2n\pi}\right]$$

$$2 = c_2^2\left[\frac{1}{2} - 0 - 0 - 0\right]$$

$$2 = \frac{c_2^2}{2}$$

$$\therefore c_2^2 = 4$$

$$\therefore c_2 = \pm 2$$

$\therefore y(x) = \pm 2 \sin n\pi x$ is the required sol^n.

2) Extremize $I(y) = \int_0^{\pi} y'^2 - y^2\, dx$ subject to $cond^n$ $\int_0^{\pi} y\, dx = 1$ and the $cond^n$ are $y(0)=0$, $y(\pi)=1$.

→ Let,

Given, $I(y) = \int_0^{\pi} y'^2 - y^2\, dx$ subject to

$J(y) = \int_0^{\pi} y\, dx$,

Here, $f(x, y, y') = y'^2 + y^2$

$g(x, y, y') = y$

Define, $f^* = f + \lambda g$

$= (y'^2 + y^2) + \lambda g$

where λ is Langrange's undetermined multiplier.

$\therefore$ To extremize I subject to $cond^n$ I

f^* must satisfy EL eq^n.

$$\frac{\partial f^*}{\partial y} - \frac{d}{dx}\left(\frac{\partial f^*}{\partial y'}\right) = 0$$

$$-2y + \lambda - \frac{d}{dx}(2y') = 0$$

$-2y - 2y''$

$-2(y'' - y$

$$y'' + y = \frac{\lambda}{2}$$

i.e. d^2 $(D^2+1)y = \frac{\lambda}{2}$

$\therefore$ Auxillary eq^n is,

$$m^2 + 1 = 0$$

$$m^2 = -1 \Rightarrow m = \pm i$$

$\therefore$ C.F. $= c_1 \cos x + c_2 \sin x$

P.I. $= \frac{1}{(D^2+1)}\left(\frac{\lambda}{2}\right)$

$= \frac{\lambda}{2} \cdot \frac{1}{(D^2+1)} e^{0x}$ $\qquad 1 = e^{0x}$

$= \frac{\lambda}{2} \cdot \frac{1}{0+1} = \frac{\lambda}{2}$

Replacing D^2 by 0 i.e. e^{0x}

$\therefore$ The general sol^n is,

$y =$ C.F. + P.I.

$y = c_1 \cos x + c_2 \sin x + \frac{\lambda}{2}$

where $y(0) = 0 \Rightarrow c_1 + \frac{\lambda}{2} = 0$ $\Rightarrow c_1 = -\lambda/2$

$y(\pi) = 1 \Rightarrow$ $\cancel{+c_1}$ $= \lambda/2$ $\quad -c_1 + \lambda/2 = 1$

$\frac{+\lambda}{2} + \frac{\lambda}{2} = 1 \Rightarrow \frac{2\lambda}{2} = 1 \Rightarrow \underline{\lambda = 1}$

To determine the value of c_2,

$$\int_0^{\pi} y \, dx = 1$$

$$\Rightarrow \int_0^{\pi} c_1 \cos x + c_2 \sin x + \frac{\lambda}{2} \, dx = 1$$

$$\Rightarrow c_1(\sin \pi - \sin 0) \cancel{+ c_2 \sin x} \ \cancel{\sin} \ H [\cos \pi - \cos 0] + \frac{\lambda}{2} x = 1$$

$$\Rightarrow c_1(0 - 0) - c_2(-1 - 1) + \frac{\lambda}{2} \cancel{x} (\pi - 0) = 1$$

$$-c_2(-2) + \frac{\lambda \pi}{2} = 1$$

$$2c_2 + \frac{\lambda \pi}{2} = 1$$

$$c_2 = \frac{\left(1 - \frac{\lambda \pi}{2}\right)}{2}$$

$$c_2 = \frac{1 - \lambda \pi}{2} \times \frac{1}{2}$$

$$\therefore c_2 = \frac{2 - \pi}{4} \qquad \because \lambda = 1$$

$$\therefore y = \frac{-\lambda}{2} \cos x + \frac{(2 - \pi)}{4} \sin x + \frac{\lambda}{2}$$

$$\therefore y = \frac{-1}{2} \cos x + \frac{(2\pi)}{4} \sin x + \frac{1}{2}$$

$$\therefore y = \frac{1}{2}\left[-\cos x + \frac{(2 - \pi)}{2} \sin x + 1\right]$$

3) Extremize $I(y) = \int_1^4 y'^2 \, dx$ subject to $\int_1^4 y \, dx = 36$

$y(1) = 3$ and $y(4) = 24$.

→ Let,

Here, $I(y) = \int_1^4 y'^2 \, dx$ & $J(y) = \int_1^4 y \, dx$

$\therefore f(x, y, y') = y'^2$ & $g(x, y, y) = y$

Define,

$$f^* = f + \lambda g$$
$$= y'^2 + \lambda y$$

where, λ is Langrange's undetermined multiplier.

The EL. eq^n is,

$$\frac{\partial f^*}{\partial y} - \frac{d}{dx}\left(\frac{\partial f^*}{\partial y'}\right) = 0$$

$$\lambda - \frac{d}{dx}(2y') = 0$$

$$2y'' = \lambda$$
$$y'' = \lambda/2$$
$$D^2 y = \lambda/2$$

Auxillary eq^n is,

$$m^2 = 0 \implies m = 0, 0$$

$$\therefore C.F. = c_1 + c_2 x$$

$$P.I. = \frac{1}{D^2} \frac{\lambda}{2}$$

$$= \frac{\lambda}{2} \frac{1}{D^2} (1) \cancel{e^{0x}}$$

$$= \frac{\lambda}{2} \frac{1}{0+1} = \frac{\lambda}{4} \frac{\lambda x^2}{4}$$

$\therefore$ The general sol^n is given by,

$$y' = C.F. + P.I.$$
$$= c_1 + c_2 x + \frac{\lambda x^2}{4}$$

Now, $y(1) = 3 \implies c_1 + c_2 + \frac{\lambda}{4} = 3$

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$4c_1 + 4c_2 + \lambda = 24$

$c_1 + 4c_2 + 4\lambda = 24$

$$Y(4) = c_1 + 4c_2 + \frac{16\lambda}{4} = c_1 + 4c_2 + 4\lambda = 24$$

$$\therefore c_1 + c_2 = 3 - \frac{\lambda}{4}$$

$$c_1 + c_2 + 3c_2 + 4\lambda = 24$$

$$3 - \frac{\lambda}{4} + 3c_2 + 4\lambda = 24$$

$$3c_2 = 24 - 4\lambda + \frac{\lambda}{4} - 3$$

$$= 21 - \frac{16\lambda + \lambda}{4}$$

$$3c_2 = 21 - \frac{15\lambda}{4}$$

$$\therefore c_2 = 7 - \frac{5\lambda}{4}$$

$$c_1 + 7 - \frac{5\lambda}{4} = 3 - \frac{\lambda}{4}$$

$$c_1 = 3 - \frac{\lambda}{4} - 7 + \frac{5\lambda}{4}$$

$$= -4 + \frac{4\lambda}{4}$$

$$= -4 + \lambda$$

$$\therefore c_1 = \lambda - 4$$

$\therefore$ To determine the λ we use

$$\int_1^4 Y \, dx = 36 \Rightarrow 36 = \int_1^4 \left(c_1 + c_2 x + \frac{\lambda x^2}{4}\right) dx$$

$$\Rightarrow 36 = \left[c_1 x + \frac{c_2 x^2}{2} + \frac{\lambda x^3}{12}\right]_1^4$$

$$36 = \left[4c_1 + 8c_2 + \frac{64\lambda}{12}\right] - \left[c_1 + \frac{c_2}{2} + \frac{\lambda}{12}\right]$$

$$= 4c_1 + 8c_2 + \frac{64\lambda}{12} - c_1 - \frac{c_2}{2} - \frac{\lambda}{12}$$

$$36 = 3c_1 + \frac{15}{2}c_2 + \frac{63\lambda}{12}$$

$$= 3\times(\lambda - 4) + \frac{15}{2}\left(7 - \frac{5\lambda}{4}\right) + \cancel{64}\,\frac{63}{12}\,\frac{21}{4}\lambda.$$

$$36 = 3\lambda - 12 + \frac{105}{2} - \frac{75\lambda}{8} + \frac{21\lambda}{4}$$

$$= 3\lambda - \frac{75\lambda}{8} + \frac{21\lambda}{4} + \frac{81}{2}$$

$$= \frac{24\lambda - 75\lambda + 42\lambda}{8} + \frac{81}{2}$$

$$36 = \frac{-9\lambda}{8} + \frac{81}{2}$$

$$36 - \frac{81}{2} =$$

$$\frac{-9}{2} = \frac{-9\lambda}{8}$$

$$\frac{-9}{2} \times \frac{8}{9} = -\lambda$$

$$-\frac{8}{2} = -\lambda$$

$$\therefore \lambda = +4$$

$$\therefore c_1 = \lambda - 4 \Rightarrow 4 - 4 = 0 \Rightarrow c_1 = 0$$

$$c_2 = 7 - \frac{5\times 4}{4} \Rightarrow 7 - 5 = 0 \Rightarrow c_2 = 2$$

$\therefore$ Required curve is,

$$y = 2x + x^2 \Rightarrow x^2 + 2x = y$$

$$2x + \frac{4x^2}{4} = 2x + x^2$$

4] Extremize $I(y(x), z(x)) = \int_0^1 y'^2 + z'^2 - 4xz' + 4z\, d$

subject to cond^n $z(1) = 1$.

$\int_0^1 y'^2 - xy' - z'^2 \, dx = 2$, $y(0) = 0$, $y(1) = 1$, $z(0) = 0$

Green's Th^m:-

Let c be a smooth simple closed curve & let D be the region enclosed by c if P & Q have continuous 1^{st} order partial derivative on D then,

$$\int_c P\,dx - Q\,dy = \int_D f(Q_x - P_y)\,dt \quad — ①$$

Application of this result is to find area of the region inclosed by D which is given by,

$$A = \iint_D dA$$

If we choose P and Q such that $Q_x - P_y = 1$ then we can use Green's Th^m ① to evaluate this area.

e.g.

1] $P = 0$ & $Q = x$ & $A = -\int_c x\,dx$

2] $P = -y$ & $Q = 0$ & $A = -\int y\,dx$

3] $P = -y/2$ & $Q = x/2$ & $A = \int_c -y/2\,dx - x/2\,dy$

$$= \frac{-1}{2}\int_c y\,dx + x\,dy$$

1] Find the plane curve of fixed perimeter that encloses maximum area.

→ Let,

$y = y(x)$ be a closed curve in a plane

of fixed parameter l.

$$\therefore \quad l = \oint ds = \oint \sqrt{1+y'^2}\, dx \quad \text{—} ①$$

The area enclosed by this curve C is given by,

$$A = \oint y \, dx \quad \text{—} ②$$

$\therefore$ The problem is to extremize eq^n ② w.r.to cond^n ①,

$$f^* = y + \lambda\sqrt{1+y'^2}$$

$\forall$ λ is Langrange's undetermined multiplier

$$\frac{\partial f^*}{\partial y} - \frac{d}{dx}\left(\frac{\partial f^*}{\partial y'}\right) = 0$$

$$1 - \frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) = 1$$

on integrating,

$$\frac{\lambda y'}{\sqrt{1+y'^2}} = x + c_1$$

$$\lambda y' = (x+c_1)\sqrt{1+y'^2}$$

squaring on both sides,

$$\lambda^2 y'^2 = (x+c_1)^2(1+y'^2)$$

$$y'^2(\lambda^2 - (x+c_1)^2) = (x+c_1)^2$$

$$y'^2 = \frac{(x+c_1)^2}{\lambda^2 - (x+c_1)^2}$$

$$y' = \frac{x+c_1}{\sqrt{\lambda^2-(x+c_1)^2}}$$

on integrating,

$$y = \int \frac{x+c_1}{\sqrt{\lambda^2-(x+c_1)^2}}$$

Put $x+c_1 = \lambda \sin t \Rightarrow dx = \lambda \cos t\, dt$ —— ③

$$\therefore y = \int \frac{\lambda \sin t}{\sqrt{\lambda^2-\lambda^2\sin^2 t}} \lambda \cos t \; dt$$

$$= \int \frac{\lambda \sin t}{\lambda\sqrt{1-\sin^2 t}} \lambda \cos t \; dt$$

$$= \int \frac{\lambda \sin t}{\lambda \cos t} \lambda \cos t \; dt$$

$$= \lambda \int \sin t \; dt$$

$$= -\lambda \cos t + c_2$$

$y - c_2 = -\lambda \cos t$ —— ④

Eliminating t from ③ & ④,

squaring eq^n ③ & ④ & adding,

$$(x+c_1)^2 + (y-c_2)^2 = \lambda^2(\sin^2 t + \cos^2 t)$$

$$(x+c_1)^2 + (y-c_2)^2 = \lambda^2$$

which is a circle with centre $(-c_1, c_2)$ & radius λ,

Perimeter of this circle is,

$2\pi\lambda$, $\therefore l = 2\pi\lambda \Rightarrow \lambda = \frac{l}{2\pi}$

eq^n becomes, $(x+c_1)^2 + (y-c_2)^2 = \frac{l^2}{4\pi^2}$

which is required curve