ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 54(1) (2024), 313-319**

CLOSURE OPERATOR AND α-IDEALS IN 0-DISTRIBUTIVE LATTICES

By
S. S. Khopade[1], S. P. Thorat[2] and Laxmi Rathour[3]
[1]Karmaveer Hire Arts, Science, Commerce and Education College, Gargoti, Maharashtra, India-416209
[2]Vivekanand College (Empowered Autonomous), Kolhapur, Maharashtra, India-416003
[3]Department of Mathematics, National Institute of Technology, Chaltlang, Aizawl, Mizoram, India-796012
Email: santajikhopade@gmail.com, thoratsanjay15@gmail.com, laxmirathour817@gmail.com
(Received: October 08, 2023*; In format: October* 29, 2023*; Revised: June* 09, 2024*;*
Accepted: June 10, 2024*)*

DOI: https://doi.org/10.58250/jnanabha.2024.54138

Abstract

A closure operator on the lattice of all ideals of a bounded 0-distributive lattice is introduced. It is observed that the ideals which are closed with respect to this closure operator are α-ideals in it and conversely.
2020 Mathematical Sciences Classification: 06D75.
Keywords and Phrases: 0- distributive lattice, ideal, closure operator, homomorphism, α-ideal

1 Introduction

As a generalization of the concept of distributive lattices on one hand and pseudocomplemented lattices on the other, 0- distributive lattices are introduced by Varlet [14]. Jayaram [6] defined and studied α-ideals in 0- distributive lattices. Additional properties of α-ideals in a 0-distributive lattice are obtained by Pawar *et al.* [8, 9]. Separation theorem for α-ideals in a 0-distributive lattice is proved in [5]. In [8], the authors have obtained a characterisation of an α-ideal using a closure operator on the lattice of all ideals of a 0-distributive lattice. The notion of closed filter in CI-algebra with some characteristic properties, is studied by Sabhapandit *et al.* [11]. Subbarayan [12] has made contributions in different aspects of 0-distributive lattices. In this paper we introduce a new closure operator on the lattice of all ideals of a 0-distributive lattice and characterise α-ideals in terms of the ideals which are closed with respect to this closure operator. Further it is observed that in a given 0- distributive lattice the ideals which are closed under this closure operator are the α-ideals in it and conversely.

2 Preliminaries

Following are some basic concepts and results needed in the sequel from references. For other non-explicitly stated elementary notions please refer to [3]. A lattice L with 0 is said to be 0 -distributive if $a \wedge b = 0$ and $a \wedge c = 0$ imply $a \wedge (b \vee c) = 0$ for any a, b, c in L. Throughout this paper L will denote a bounded 0-distributive lattice unless otherwise specified. For a lattice $L, \mathcal{I}(L)$ denotes the set of all ideals of L. Then $(\mathcal{I}(L), \wedge, \vee)$ is a lattice where $I \wedge J = I \cap J$ and $I \vee J = (I \cup J]$, for any two ideals I and J of L. For any non-empty subset A of L, define $A^* = \{x \in L : x \wedge a = 0, \text{ for each a} \in \text{A}\}$. By A^{**} we mean $(A^*)^*$. Note that when $A = \{a\}$ then $A^* = (a]^*$ and also denoted by $(a)^*$. An ideal I in L is called an annihilator ideal if $I = A^*$, for a non-empty subset A of L. Let L and L' denote bounded 0-distributive lattices and $f : L \to L'$ be a homomorphism. f is called an annihilator preserving homomorphism if $f(A^*) = \{f(A)\}^*$ for any non-empty subset A of L. An ideal I of L is called an α-ideal if $\{x\}^{**} \subseteq I$ for each $x \in I$. Closure operator on L is a mapping $f : L \to L$ satisfying the following conditions: (i) $x \leq f(x)$, (ii) $x \leq y \Rightarrow f(x) \leq f(y)$ and (iii) $f(f(x)) = f(x)$.

Result 2.1. (Varlet [14]). *A lattice L with 0 is 0 - distributive if and only if A^* is an ideal for any non-empty subset A of L.*

Following result can be proved easily.

Result 2.2. *In a 0-distributive lattice L, for all $a, b, c \in L$ we have*

(i) $\{a\}^{**} \cap \{b\}^{**} = \{a \wedge b\}^{**}$.
(ii) $\{a\}^{*} \cap \{b\}^{*} = \{a \vee b\}^{*}$.
(iii) $\{a\}^{**} = \{b\}^{**} \Rightarrow \{a \wedge c\}^{**} = \{b \wedge c\}^{**}$.

Result 2.3. (Pawar and Mane [8]). *In a bounded 0-distributive lattice L following statements are equivalent.*
(i) *For* $x, y \in L, \{x\}^* = \{y\}^*, x \in I \Rightarrow y \in I$.
(ii) $I = U\{\{x\}^{**} : x \in I\}$.
(iii) *For* $x, y \in L,\ \mathrm{h}(x) = h(y), x \in I \Rightarrow y \in I$, *where* $\mathrm{h}(x) = \{M : M$ *is a minimal prime ideal containing* $x\}$.
(iv) I *is an* α*-ideal.*

Result 2.4. (Jayaram [5]). *Let* L *be a 0-distributive lattice. Let* I *be an* α*-ideal and* S *be a meet sub semi lattice of* L *such that* $I \cap S = \emptyset$. *Then there exists a prime* α*-ideal* P *in* L *containing* I *and disjoint with* S.

Result 2.5. (Pawar and Mane [8]). *Every annihilator ideal in a 0-distributive lattice* L *is an* α*-ideal.*

Result 2.6. (Pawar and Khopade [9]). *Let* L *and* L' *be any two bounded 0-distributive lattices and let* $f : L \to L'$ *be an annihilator preserving onto homomorphism, Then*
(i) *If* I *is an* α*-ideal of* L, *then* $f(I)$ *is an* α*-ideal of* L'.
(ii) *If* I' *is an* α*-ideal of* L', *then* $f^{-1}(I')$ *is an* α*-ideal of* L.

3 Closure operator

In this section we introduce a closure operator on $\mathcal{I}(L)$.

Define $\mathcal{B}(L) = \{\{a\}^{**} : \mathrm{a} \in L\}$. L being 0-distributive lattice, $\mathcal{B}(L) \subseteq \mathcal{I}(L)$ (by Result 2.1) but $\mathcal{B}(L)$ is not necessarily a sub lattice of the lattice $\mathcal{I}(L)$. For this consider the following example.

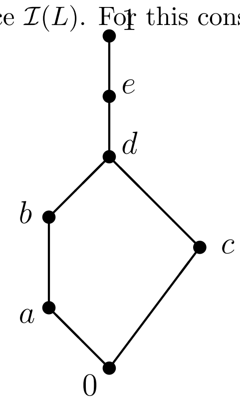

Figure **3.1**

Example 3.1. Consider the bounded 0 - distributive lattice $L = \{0$, a, b, c, d, e, $1\}$ as shown by the Hasse Diagramme in Figure 3.1. Here $\{a\}^{**} = \{0, a, b\}$ and $\{c\}^{**} = \{0, c\}$. Hence $\{a\}^{**} \vee \{c\}^{**} = \{0, a, b, c, d\} \notin \mathcal{B}(L)$. Hence the set $\mathcal{B}(L)$ is a poset under set inclusion but need not be a sub lattice of the lattice $\mathcal{I}(L)$.

For $\{a\}^{**}, \{b\}^{**} \in \mathcal{B}(L)$. Define $\{a\}^{**} \sqcap \{b\}^{**} = \{a \wedge b\}^{**}$ and $\{a\}^{**} \sqcup \{b\}^{**} = \{a \vee b\}^{**}$. Then we have

Theorem 3.1. $(\mathcal{B}(L), \sqcap, \sqcup)$ *is a bounded lattice.*

Proof. Obviously, $\{a \wedge b\}^{**}$ is the infimum of $\{a\}^{**}$ and $\{b\}^{**}$ in $(\mathcal{B}(L), \subseteq)$. To prove $\{a \vee b\}^{**}$ is the supremum of $\{a\}^{**}$ and $\{b\}^{**}$ in $(\mathcal{B}(L), \subseteq)$.$\{a \vee b\}^{**}$ is an upper bound of $\{a\}^{**}$ and $\{b\}^{**}$ in $(\mathcal{B}(L), \subseteq)$. Let $\{c\}^{**}$ be any other upper bound of $\{a\}^{**}$ and $\{b\}^{**}$ in $(\mathcal{B}(L), \subseteq)$. Let $\mathrm{t} \in \{a \vee b\}^{**}$. Then $(t] \cap \{a \vee b\}^{*} = \{0\}$. By Result 2.2 (ii) we get $(t] \cap [\{a\}^{*} \cap \{b\}^{*}] = \{0\}$, which implies $(t] \cap \{a\}^{*} \subseteq \{b\}^{**}$. But as $\{b\}^{**} \subseteq \{c\}^{**}$ we get $(t] \cap \{a\}^{*} \subseteq \{c\}^{**}$. Thus $(t] \cap \{a\}^{*} \cap \{c\}^{*} = \{0\}$, implies $(t] \cap \{c\}^{*} \subseteq \{a\}^{**}$. Again, as $\{a\}^{**} \subseteq \{c\}^{**}$, we get $(t] \cap \{c\}^{*} \subseteq \{c\}^{**}$, that is $(t] \cap \{c\}^{*} = \{0\}$. Therefore $(t] \subseteq \{c\}^{**}$ which yields $\mathrm{t} \in \{c\}^{**}$. This shows that $\{a \vee b\}^{**} \subseteq \{c\}^{**}$ and hence $\{a \vee b\}^{**}$ is the supremum of $\{a\}^{**}$ and $\{b\}^{**}$ in $(\mathcal{B}(L), \subseteq)$. As $\{0\}^{**} = \{0\}$ and $\{1\}^{**} = L$ belong to $\mathcal{B}(L), (\mathcal{B}(L), \sqcap, \sqcup)$ is a bounded lattice. $\square$

Corollary 3.1. *The lattice* $(\mathcal{B}(L), \sqcap, \sqcup)$ *is a homomorphic image of the lattice* L.

Proof. Define $\theta : L \to \mathcal{B}(L)$ by $\theta(a) = \{a\}^{**}$ for each $a \in L$. Then $\theta(a \wedge b) = \{a \wedge b\}^{**} = \{a\}^{**} \sqcap \{b\}^{**} = \theta(a) \sqcap \theta(b)$ and $\theta(a \vee b) = \{a \vee b\}^{**} = \{a\}^{**} \sqcup \{b\}^{**} = \theta(a) \sqcup \theta(b)$ hold for all $a, b \in L$. Hence θ is a homomorphism. As θ is onto, the result follows. $\square$

Remark 3.1. *Note that the homomorphism θ is not necessarily one-one. For this consider the 0 - distributive lattice in Example 3.1. Here for $a \neq b$ in L we have $\{a\}^{**} = \{b\}^{**}$.*

For any ideal I of L, define $\delta(I) = \{\{a\}^{**} : a \in I\}$ and for any ideal $\bar{I}$ of $\mathcal{B}(L)$, define $\overleftarrow{\delta}(\bar{I}) = \left\{a \in L : \{a\}^{**} \in \bar{I}\right\}$. With these notations we prove

Theorem 3.2.

(i) $\delta(I)$ is an ideal of $\mathcal{B}(L)$, for any ideal I of L.

(ii) $\overleftarrow{\delta}(\bar{I})$ is an ideal of L, for any ideal $\bar{I}$ of $\mathcal{B}(L)$.

(iii) For any two ideals I and J of $L, I \subseteq J \Rightarrow \delta(I) \subseteq \delta(J)$.

(iv) For any two ideals $\bar{I}$ and $\bar{J}$ of $\mathfrak{B}(L), \bar{I} \subseteq \bar{J} \Rightarrow \overleftarrow{\delta}(\bar{I}) \subseteq \overleftarrow{\delta}(\bar{J})$.

Proof. (i). Let I be any ideal of L. As $0 \in I, \{0\}^{**} = \{0\} \in \delta(I)$. Hence $\delta(I)$ is non empty. Let $\{a\}^{**}, \{b\}^{**} \in \mathcal{B}(L)$ such that $\{a\}^{**} \subseteq \{b\}^{**}$ and $\{b\}^{**} \in \delta(I)$. Then $\{b\}^{**} = \{x\}^{**}$ for some $x \in I$. Thus $\{a\}^{**} = \{a\}^{**} \sqcap \{b\}^{**} = \{a\}^{**} \sqcap \{x\}^{**} = \{a \wedge x\}^{**}$. As $a \wedge x \in I$, we get $\{a\}^{**} \in \delta(I)$. Let $\{a\}^{**}, \{b\}^{**} \in \delta(I)$. Therefore $\{a\}^{**} = \{x\}^{**}$ and $\{b\}^{**} = \{y\}^{**}$ for some $x, y \in I$. Hence $\{a\}^{**} \sqcup \{b\}^{**} = \{x\}^{**} \sqcup \{y\}^{**} = \{x \vee y\}^{**}$. As $x \vee y \in I$, we get $\{x \vee y\}^{**} \in \delta(I)$ Hence $\{a\}^{**} \sqcup \{b\}^{**} \in \delta(I)$. Therefore $\delta(I)$ is an ideal of $\mathcal{B}(L)$.

(ii) Let $\bar{I}$ be any ideal of $\mathcal{B}(L) \cdot \{0\}^{**} = \{0\} \in \bar{I}$ implies $0 \in \overleftarrow{\delta}(\bar{I})$. Hence $\overleftarrow{\delta}(\bar{I})$ is non-empty. Let $a, b \in$ L such that $a \leqq b$ and $b \in \overleftarrow{\delta}(\bar{I})$. Then $\{a\}^{**} \subseteq \{b\}^{**}$ and $\{b\}^{**} \in \bar{I}$. $\bar{I}$ being an ideal we get $\{a\}^{**} \in \bar{I}$. But then $a \in \overleftarrow{\delta}(\bar{I})$. Let $a, b \in \overleftarrow{\delta}(\bar{I})$. Then $\{a\}^{**}, \{b\}^{**} \in \bar{I}$ implies $\{a\}^{**} \sqcup \{b\}^{**} = \{a \vee b\}^{**} \in \bar{I}$. Therefore $a \vee b \in \overleftarrow{\delta}(\bar{I})$. This proves $\overleftarrow{\delta}(\bar{I})$ is an ideal of L.

(iii) Let I and J be two ideals of L such that $I \subseteq J$. Let $\{a\}^{**} \in \delta(I)$. Then $\{a\}^{**} = \{x\}^{**}$ for some $x \in I$. But then, since $I \subseteq J$, we get $x \in J$. This in turns gives $\{a\}^{**} \in \delta(J)$. Hence $\delta(I) \subseteq \delta(J)$.

(iv) Let $\bar{I}$ and $\bar{J}$ be any two ideals of $B(L)$ such that $\bar{I} \subseteq \bar{J}$. Let $x \in \overleftarrow{\delta}(\bar{I})$. Then $\{x\}^{**} \in \bar{I}$ implies $\{x\}^{**} \in \bar{J}$. Hence $x \in \overleftarrow{\delta}(\bar{J})$ and the result follows. $\square$

As $\delta(I)$ is an ideal of $\mathcal{B}(L)$, for any ideal I of L, we have the mapping $\delta : \mathcal{I}(L) \to \mathcal{I}(\mathcal{B}(L))$ is well defined, where $\mathcal{I}(\mathcal{B}(L))$ denotes the lattice of all ideals of the lattice $\mathcal{B}(L)$. Further we have

Theorem 3.3. $\delta : \mathcal{I}(L) \to \mathcal{I}(\mathcal{B}(L))$ *is a* $\{0, 1\}$ *homomorphism.*

Proof. Let I and J be any ideals in $\mathcal{I}(L)$. $\delta(I \cap J) \subseteq \delta(I) \cap \delta(J)$ (by Theorem 3.2 - (iii)). Let $\{a\}^{**} \in \delta(I) \cap \delta(J)$. Then $\{a\}^{**} \in \delta(I)$ implies $\{a\}^{**} = \{i\}^{**}$ for some $i \in I$ and $\{a\}^{**} \in \delta(J)$ gives $\{a\}^{**} = \{j\}^{**}$ for some $j \in J$. Thus $\{a\}^{**} = \{i\}^{**} \sqcap \{j\}^{**} = \{i \wedge j\}^{**}$. As $i \wedge j \in I \cap J$, we get $\{a\}^{**} \in \delta(I \cap J)$. This shows that $\delta(I) \cap \delta(J) \subseteq \delta(I \cap J)$. Combining both the inclusions we get $\delta(I \cap J) = \delta(I) \cap \delta(J)$.

Now, again by Theorem 3.2 - (iii), $\delta(I) \vee \delta(J) \subseteq \delta(I \vee J)$. Let $\{a\}^{**} \in \delta(I \vee J)$. Hence $\{a\}^{**} = \{y\}^{**}$ for some $y \in I \vee J$. Therefore $y \leq i \vee j$ for some $i \in I$ and $j \in J$. This yields $\{y\}^{**} \subseteq \{i \vee j\}^{**} = \{i\}^{**} \sqcup \{j\}^{**}$. Therefore $\{a\}^{**} = \{y\}^{**} \in \delta(I) \vee \delta(J)$. Hence $\delta(I \vee J) \subseteq \delta(I) \vee \delta(J)$. Combining both the inclusions we get $\delta(I \vee J) = \delta(I) \vee \delta(J)$.

This proves that $\delta : \mathcal{I}(\mathrm{L}) \to \mathcal{I}(\mathcal{B}(L))$ is a homomorphism. Again $\delta((0]) = \{\{0\}^{**}\} = \{\{0\}\}$ and $\delta((1]) = \{\{1\}^{**}\} = \{L\}$, shows δ is a $\{0, 1\}$ homomorphism. $\square$

By Theorem 3.2, we get two mappings $\delta : \mathcal{I}(L) \to \mathcal{I}(\mathcal{B}(L))$ and $\overleftarrow{\delta} : \mathcal{I}(\mathcal{B}(L)) \to \mathcal{I}(L)$. Hence $\delta \circ \overleftarrow{\delta} : \mathcal{I}(\mathcal{B}(L)) \to \mathcal{I}(\mathcal{B}(L))$ and $\overleftarrow{\delta} \circ \delta : \mathcal{I}(L) \to \mathcal{I}(L)$. About these two mappings we have

Theorem 3.4.

(i) $\delta \circ \overleftarrow{\delta}$ is an identity mapping on $\mathcal{I}(\mathcal{B}(L))$.

(ii) $\overleftarrow{\delta} \circ \delta$ is a closure operator on $\mathcal{I}(L)$.

Proof. (i) Let $\bar{I}$ be any ideal of $\mathcal{B}(L)$. Let $\{x\}^{**} \in \delta \circ \overleftarrow{\delta}(\bar{I}) = \delta(\overleftarrow{\delta}(\bar{I}))$. Hence $\{x\}^{**} = \{y\}^{**}$ for some $y \in \overleftarrow{\delta}(\bar{I})$. But then $\{y\}^{**} \in \bar{I}$, which implies $\{x\}^{**} \in \bar{I}$. This gives $\delta \circ \overleftarrow{\delta}(\bar{I}) \subseteq \bar{I}$. Conversely, let $\{x\}^{**} \in \bar{I}$. Then $x \in \overleftarrow{\delta}(\bar{I})$ and consequently $\{x\}^{**} \in \delta(\overleftarrow{\delta}(\bar{I}))$ (since $\overleftarrow{\delta}(\bar{I})$ is an ideal of L). Hence $\bar{I} \subseteq \delta \circ \overleftarrow{\delta}(\bar{I})$. From both the inclusions we get $\delta \circ \overleftarrow{\delta}(\bar{I}) = \bar{I}$. Hence $\delta \circ \overleftarrow{\delta}$ is an identity mapping on $\mathcal{I}(\mathcal{B}(L))$.

(ii) Let $I \in \mathcal{I}(L)$ and $x \in I$. Then $\{x\}^{**} \in \delta(I)$ and by Theorem 3.2 - (i), $\delta(I)$ is an ideal of $\mathcal{B}(L)$, which yields $x \in \overleftarrow{\delta} \circ \delta(I)$. Hence $I \subseteq \overleftarrow{\delta} \circ \delta(I)$. (3.1)

Let $I, J \in \mathcal{I}(L)$ and $I \subseteq J$. As δ and $\overleftarrow{\delta}$ are isotone mappings (by Theorem 3.2),
we get $\overleftarrow{\delta} \circ \delta(I) \subseteq \overleftarrow{\delta} \circ \delta(J)$. (3.2)

Finally, let $I \in \mathcal{I}(L)$. As $I \subseteq \overleftarrow{\delta} \circ\delta(I)$, applying (3.2) we get $\overleftarrow{\delta} \circ\delta(I) \subseteq \overleftarrow{\delta} \circ\delta\left(\overleftarrow{\delta} \circ\delta(I)\right)$. Conversely, let $x \in \overleftarrow{\delta} \circ\delta\left(\overleftarrow{\delta} \circ \delta(I)\right)$. Then $\{x\}^{**} \in \delta\left(\overleftarrow{\delta} \circ \delta(I)\right)$ implies $\{x\}^{**} = \{y\}^{**}$ for some $y \in \overleftarrow{\delta} \circ \delta(I)$. But then $\{y\}^{**} \in \delta(I)$, which implies $\{x\}^{**} \in \delta(I)$. This gives $x \in \overleftarrow{\delta} \circ \delta(I)$. This proves $\overleftarrow{\delta} \circ\delta\left(\overleftarrow{\delta} \circ\delta(I)\right) \subseteq \overleftarrow{\delta} \circ\delta(I)$. Combining both the inclusions we get $\overleftarrow{\delta} \circ\delta\left(\overleftarrow{\delta} \circ \delta(I)\right) = \overleftarrow{\delta} \circ\delta(I)$. (3.3)

From (3.1), (3.2) and (3.3) we get $\overleftarrow{\delta} \circ \delta$ is a closure operator on $\mathcal{I}(L)$. □

Remark 3.2. *The mapping $\delta : \mathcal{I}(L) \to \mathcal{I}(\mathcal{B}(L))$ is a homomorphism follows from Theorem* 3.3. *Let $\bar{I}$ be any ideal of $\mathcal{B}(L)$. As $\overleftarrow{\delta}(\bar{I})$ is an ideal of L and $\delta \circ \overleftarrow{\delta}(\bar{I}) = \bar{I}$, we get the mapping $\delta : \mathcal{I}(L) \to \mathcal{I}(\mathcal{B}(L))$ is onto. Hence the lattice $\mathcal{I}(\mathcal{B}(L))$ is a homomorphic image of the lattice $\mathcal{I}(L)$.*

4 α - ideals

In this section we show that the ideals in L which are closed with respect to the closure operator $\overleftarrow{\delta} \circ\delta$ defined on $\mathcal{I}(L)$ are α-ideals in L and conversely. Let $\mathcal{C}(L)$ denote the set of all ideals in L which are closed with respect to the closure operator $\overleftarrow{\delta} \circ\delta$ defined on $\mathcal{I}(L)$. Thus $\mathcal{C}(L) = \left\{I \in \mathcal{I}(L) : \overleftarrow{\delta} \circ \delta(I) = I\right\}$. Obviously, $(0]$ and $(1]$ belong to $\mathcal{C}(L)$. Hence $\mathcal{C}(L)$ is a non-empty subset of $\mathcal{I}(L)$ but not necessarily a sublattice of the lattice $\mathcal{I}$ (L). This follows by the 0-distributive lattice given in Example 3.1. Here $\mathcal{C}(L) = \{(0], (b], (c]\}$ and $(b] \vee (c] = (d]$. As $(d] \notin \mathcal{C}(L)$, the subset $\mathcal{C}(L)$ is not a sublattice of the lattice $\mathcal{I}(L)$. Though $\mathcal{C}(L)$ does not form a sublattice of the lattice $\mathcal{I}(L)$, it forms a lattice on its own. This we prove in the following theorem.

Theorem 4.1. *$(\mathcal{C}(L), \overline{\wedge}, \underline{\vee})$ is a bounded lattice where $\overline{\wedge}$ and $\underline{\vee}$ are defined by $I \,\overline{\wedge}\, J = I \cap J$ and $I \,\underline{\vee}\, J = \overleftarrow{\delta} \circ \delta(I \vee J)$ for $I, J \in \mathcal{C}(L)$*

Proof. (i) First we prove that for $I, J \in \mathcal{C}(L), I \cap J \in \mathcal{C}(L)$. As $\overleftarrow{\delta}$ and δ are isotone mappings, we get $\overleftarrow{\delta} \circ \delta$ is also isotone. Hence $\overleftarrow{\delta} \circ \delta(I \cap J) \subseteq \overleftarrow{\delta} \circ \delta(I) \cap \overleftarrow{\delta} \circ \delta(J)$.

Let $x \in \overleftarrow{\delta} \circ \delta(I) \cap \overleftarrow{\delta} \circ \delta(J)$. Then $\{x\}^{**} \in \delta(I) \cap \delta(J) = \delta(I \cap J)$. This gives $x \in \overleftarrow{\delta} \circ \delta(I \cap J)$. Hence $\overleftarrow{\delta} \circ\delta(I) \cap \overleftarrow{\delta} \circ\delta(J) \subseteq \overleftarrow{\delta} \circ\delta(I \cap J)$. Combining both the inclusions we get $\overleftarrow{\delta} \circ\delta(I \cap J) = \overleftarrow{\delta} \circ\delta(I) \cap \overleftarrow{\delta} \circ\delta(J) = I \cap J$ (since $I, J \in \mathcal{C}(L)$). This proves $I \cap J \in \mathcal{C}(L)$. Thus the infimum of $I, J \in \mathcal{C}(L)$ is $I \cap J$. Hence $I \overline{\wedge} J = I \cap J$.

(ii) First note that, by Theorem 3.4 - (ii), $\overleftarrow{\delta} \circ \delta(I) \in \mathcal{C}(L)$, for any ideal I of L. Let $I, J \in \mathcal{C}(L)$. Then $I = \overleftarrow{\delta} \circ \delta(I) \subseteq \overleftarrow{\delta} \circ \delta(I \vee J)$ and $J = \overleftarrow{\delta} \circ \delta(J) \subseteq \overleftarrow{\delta} \circ \delta(I \vee J)$ (since $\overleftarrow{\delta} \circ \delta$ is isotone). Thus $\overleftarrow{\delta} \circ \delta(I \vee J)$ is an upper bound of I and J in $\mathcal{C}(L)$. Let $K \in \mathcal{C}(L)$, such that $I \subseteq K$ and $J \subseteq K$. Then $I \vee J \subseteq K$ implies $\overleftarrow{\delta} \circ \delta(I \vee J) \subseteq \overleftarrow{\delta} \circ \delta(K) = K$ (since $K \in \mathcal{C}(L)$). This shows that $\overleftarrow{\delta} \circ \delta(I \vee J)$ is the supremum of I and J in $\mathcal{C}(L)$ i. e. $I \underline{\vee} J = \overleftarrow{\delta} \circ \delta(I \vee J)$. As $(0] \in \mathcal{C}(L)$ and $L \in \mathcal{C}(L)$, $(\mathcal{C}(L), \overline{\wedge}, \underline{\vee})$ is a bounded lattice. □

We know that the lattice $\mathcal{I}(\mathcal{B}(L))$ is a homomorphic image of the lattice $\mathcal{I}(L)$ (see Remark 3.2). But interestingly we have

Theorem 4.2. *The lattice $\mathcal{C}(L)$ is isomorphic with the lattice $\mathcal{I}(\mathcal{B}(L))$.*

Proof. Define the mapping $\psi : \mathcal{C}(L) \to \mathcal{I}(B(L))$ by $\psi(I) = \delta(I)$ for each $I \in \mathcal{C}(L)$, which is clearly a well defined mapping.
(i) Let $\psi(I) = \psi(J)$ for $I, J \in \mathcal{C}(L)$. Then we have $\delta(I) = \delta(J)$. Therefore $\overleftarrow{\delta} \circ\delta(I) = \overleftarrow{\delta} \circ\delta(J)$ which implies $I = J$ (since $I, J \in \mathcal{C}(L)$). This shows that ψ is one-one.

(ii) Let $\bar{I}$ be any ideal of $\mathcal{B}(L)$. Then $\overleftarrow{\delta}(\bar{I})$ is an ideal of L (by Theorem 3.2 - (ii)) and $\delta \circ \overleftarrow{\delta}(\bar{I}) = \bar{I}$ (by Theorem 3.4 - (i)). Then $\overleftarrow{\delta} \circ \delta(\overleftarrow{\delta}(\bar{I})) = \overleftarrow{\delta}(\delta(\overleftarrow{\delta}(\bar{I}))) = \overleftarrow{\delta}(\delta \circ \overleftarrow{\delta}(\bar{I})) = \overleftarrow{\delta}(\bar{I})$. This shows that $\overleftarrow{\delta}(\bar{I}) \in \mathcal{C}(L)$. As $\psi(\overleftarrow{\delta}(\bar{I})) = \delta(\overleftarrow{\delta}(\bar{I})) = \delta \circ \overleftarrow{\delta}(\bar{I}) = \bar{I}$, we get ψ is onto.
(iii) Let $I, J \in \mathcal{C}(L)$. Then by definition of ψ and by Theorem 3.3 we get, $\psi(I \bar{\wedge} J) = \psi(I \cap J) = \delta(I \cap J)$ $= \delta(I) \cap \delta(J) = \psi(I) \cap \psi(J)$. And by definition of $\veebar$ in $\mathcal{C}(L)$ we get $\psi(I \veebar J) = \delta(I \veebar J) = \delta\left(\overleftarrow{\delta} \circ \delta(I \vee J)\right)$ $= \delta(I \vee J)$ (since $\delta \circ \overleftarrow{\delta}$ is an identity map). Thus $\psi(I \veebar J) = \delta(I \vee J) = \delta(I) \vee \delta(J) = \psi(I) \vee \psi(J)$. This proves that ψ is a homomorphism. From (i) - (iii) we get ψ is an isomorphism. ☐

Following theorem gives a necessary and sufficient conditions for an ideal I of L to be a member of $\mathcal{C}(L)$.

Theorem 4.3. *For any ideal I of L, following statements are equivalent.*

(i) $I \in \mathcal{C}(L)$.
(ii) *For* $x, y \in L, \{x\}^{**} = \{y\}^{**}, x \in I \Rightarrow y \in I$.
(iii) *For* $x, y \in L, \{x\}^{*} = \{y\}^{*} \quad x \in I \Rightarrow y \in I$.
(iv) $I = \cup\{\{x\}^{**} : x \in I\}$.
(v) *For* $x, y \in L$, $h(x) = h(y)$, $x \in I \Rightarrow y \in I$, *where* $h(x) = \{M : M$ *is a minimal prime ideal containing* $x\}$.
(vi) I *is an* α*-ideal.*

Proof. The equivalence of the statements (iii) to (vi) follows by Result 2.3.

(ii) $\Leftrightarrow$ (iii): As $\{x\}^{**} = \{y\}^{**} \Leftrightarrow \{x\}^{*} = \{y\}^{*}$ for any $x, y \in L$, the equivalence follows.

(i) $\Rightarrow$ (ii): Let $I \in \mathcal{C}(L)$. Let $x, y \in L$ such that $\{x\}^{**} = \{y\}^{**}$ and $x \in I$. As $x \in I$, we have $\{x\}^{**} \in \delta(I)$. But then, by assumption, we get $\{y\}^{**} \in \delta(I)$. This gives $y \in \overleftarrow{\delta} \circ \delta(I)$. Again by assumption that $I \in \mathcal{C}(L)$, we get $y \in I$. Thus the implication follows.

(ii) $\Rightarrow$ (i): Let $I \in \mathcal{I}(L)$ satisfying condition in (ii). By Theorem 3.4, we have $I \subseteq \overleftarrow{\delta} \circ \delta(I)$. To prove $\overleftarrow{\delta} \circ \delta(I) \subseteq I$. On contrary assume that $\overleftarrow{\delta} \circ \delta(I) \nsubseteq I$. Then there exists $x \in \overleftarrow{\delta} \circ \delta(I)$ such that $x \notin I$. Then $\{x\}^{**} \in \delta(I)$ which implies $\{x\}^{**} = \{y\}^{**}$ for some $y \in I$. But then, by assumption, $x \in I$; a contradiction. Hence $\overleftarrow{\delta} \circ \delta(I) \subseteq I$. Combining both the inclusions, we get $\overleftarrow{\delta} \circ \delta(I) = I$. Hence $I \in \mathcal{C}(L)$ and the implication follows. Hence all the statements are equivalent. ☐

Using the property that $I \in \mathcal{C}(L)$ if and only if I is an α-ideal, proved in above theorem, we get

Corollary 4.1. $(a] \in \mathcal{C}(L)$ *if and only if* $(a] = \{a\}^{**}$ *for any* $a \in L$.

Proof. Let $(a] \in \mathcal{C}(L)$. Then by Theorem 4.3, $(a]$ is an α-ideal of L. This gives $\{a\}^{**} \subseteq (a]$ (by definition of α-ideal). As we obviously have (a] $\subseteq \{a\}^{**}$, the proof of if part follows. Conversely, suppose $(a] = \{a\}^{**}$. We know that every annihilator ideal is an α-ideal, therefore $\{a\}^{**} = (a]$ is an α-ideal. Thus again by Theorem 4.3, we get $(a] \in \mathcal{C}(L)$. ☐

$I^{*} \in \mathcal{C}(L)$ for any ideal I in L, because I^{*} is an α-ideal of L (see Result 2.5). Hence we have

Corollary 4.2. *The lattice* $(\mathcal{C}(L), \bar{\wedge}, \veebar)$ *is a pseudo complemented lattice.*

Define $A_0(L) = \{\{x\}^{*} : x \in L\}$. Then $\left(A_0(L), \widehat{\wedge}, \widetilde{\vee}\right)$ is a lattice, where $\{x\}^{*}\widehat{\wedge}\{y\}^{*} = \{x \vee y\}^{*}$ and $\{x\}^{*}\widetilde{\vee}\{y\}^{*} = \{x \wedge y\}^{*}$. This lattice is called as a lattice of all annulets of L. For any ideal I in L, the set $\{\{x\}^{*} : x \in I\}$ is a filter in $A_0(L)$ and for any filter F in $A_0(L)$, the set $\{x \in L : \{x\}^{*} \in F\}$ is an ideal of L. Let $\mathcal{F}(A_0(L))$ denote the lattice of all filters in $A_0(L)$. Then the maps $\alpha : \mathcal{I}(L) \to \mathcal{F}(A_0(L))$ defined by $\alpha(I) = \{\{x\}^{*} : x \in I\}$ and $\beta : \mathcal{F}(A_0(L)) \to \mathcal{I}(L)$ defined by $\beta(F) = \{x \in L : \{x\}^{*} \in F\}$ are well defined isotone maps.

We need the following results from [8]:

Lemma 4.1. ([8], Theorem 9). *The map* $\beta \circ \alpha : \mathcal{I}(L) \to \mathcal{I}(L)$ *is a closure operator on* $\mathcal{I}(L)$.

Lemma 4.2. ([8] Theorem 10).
For any ideal I in L, following statements are equivalent.

(i) I *is an* α*-ideal.*

(ii) $\beta \circ \alpha(I) = I$.

Using above two lemmas and Theorem 4.3 we get
$\mathcal{C}(L) = \left\{ I \in \mathcal{I}(L) : \overleftarrow{\delta} \circ \delta(I) = I \right\} = \{I \in \mathcal{I}(L) : \beta \circ \alpha(I) = I\}$. Hence an ideal I in L is closed with respect to the closure operator $\overleftarrow{\delta} \circ \delta$ if and only if it is closed with respect to the closure operator $\beta \circ \alpha$ defined on $\mathcal{I}(L)$. Thus we have

Corollary 4.3. *For any ideal I of L, $\overleftarrow{\delta} \circ \delta(I) = I$ if and only if $\beta \circ \alpha(I) = I$.*

Let I be an ideal of L. If there exists a prime ideal P of L such that $I \subseteq P$ and P is minimal in the class of all prime ideals containing I, then P is called a prime ideal belonging to I. We know that any prime ideal of L need not be an α-ideal. For this consider the lattice $L = \{0, a, b, c, d, e, 1\}$ whose Hasse diagram is as in Figure 3.1. The ideal $(e]$ is a prime ideal but not an α-ideal. For, $d \in (e]$ but $(d]^{**} = L \nsubseteq (e]$.

In the following theorem we show that a prime ideal belonging to an α-ideal is an α-ideal.

Theorem 4.4. *Let I be an α-ideal of L. Let P be a prime ideal belonging to I, then P is an α ideal.*

Proof. Suppose P is not an α-ideal. Hence there exist x, y in L such that $\{x\}^{**} = \{y\}^{**}, x \in P$ but $y \notin P$ (see Theorem 4.3). Consider the filter $F = (L \setminus P) \vee [x \wedge y)$. Claim that $F \cap I = \emptyset$. Let $F \cap I \neq \emptyset$. Select $a \in F \cap I$. Then $a \in F$ implies $a \geq r \wedge s$ for some $r \in (L \setminus P)$ and $s \geq x \wedge y$. But then $a \geq r \wedge x \wedge y$ and therefore $r \wedge x \wedge y \in I$ (as $a \in I$). Since $\{x\}^{**} = \{y\}^{**}$, using the Result 2.2, we get $\{r \wedge x\}^{**} = \{r \wedge y\}^{**}$ and hence $\{r \wedge x \wedge y\}^{**} = \{r \wedge y\}^{**}$. Since $r \wedge x \wedge y \in I$ and I is an α-ideal, by Theorem 4.3, we get $r \wedge y \in I$. Hence $r \wedge y \in P$ (since $I \subseteq P$). Now $r \wedge y \in P$, P is a prime ideal and $r \notin P$ imply $y \in P$; which contradicts our assumption. Hence we must have $F \cap I = \emptyset$. Therefore, by Result 2.4, there exists a prime ideal Q containing I and disjoint with F. Thus $Q \subseteq P$. Moreover $F \cap Q = \emptyset$ and $x \wedge y \in F$ implies $x \wedge y \notin Q$. Hence $Q \neq P$ (since $x \in P \Rightarrow x \wedge y \in P$) i. e. $Q \subset P$. But this contradicts to the fact that P is minimal in the class of all prime ideals containing I. Hence we must have P is an α-ideal. ☐

Making an appeal to Theorem 4.1, Theorem 4.3 and Result 2.6, we establish

Corollary 4.4. *Let L and L' be bounded 0- distributive lattices and let $f : L \to L'$ be an annihilator preserving onto homomorphism. Then we have*
(i) If $I \in \mathcal{C}(L)$, then $f(I) \in \mathcal{C}(L')$.
(ii) If $I' \in \mathcal{C}(L')$, then $f^{-1}(I') \in \mathcal{C}(L)$.

5 Conclusion

The present investigation provides a new way to define closure operator on the lattice of all ideals of a bounded 0 - distributive lattice. Moreover the ideals closed with respect to this closure operator are $\alpha-$ ideals. Therefore this work will motivate and useful to study more properties of $\alpha-$ ideals.

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