A

PROJECT REPORT ON

METRIC SPACES AND THEIR GENERALIZATIONS: EXPLORING CONCEPTS OF DISTANCE AND BEYOND

SUBMITTED TO

DEPARTMENT OF SCIENCE AND TECHNOLOGY

SESSION 2022

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Project Completion Certificate

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9. Broad Subject Area of Research Project:
10. Project Duration (in weeks): 8
11. Date of Start of Research Project: 01 June 2020
12. Date of Completion of Research Project: 30 July 2020
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Acknowledgement

I would like to express my special thanks of gratitude to my mentor and department of science to provide me this opportunity, and I really thankful for guide me as a inspire mentorship project and I would also thankful for DST Inspire division for provide me financial support.

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INTRODUCTION

In 1906 Maurice Frechet introduced metric spaces in his work sur quelques points ducalcul fonctionnel.However the name is due to Felix Hausdorff.Let X be an arbitrary set,which could consist of vectors in $\mathbb{R}^n$, functions, sequences,matrices,etc.We want to endow this set with a metric;i.e a way to measure distances between elements of X.A distance or metric is a function d: $X \times X \rightarrow \mathbb{R}$ such that if we take two elements x,y $\in$ X the number d(x.y) gives us the distance between them.However, not just any function may be considered a metric: as we will see in the formal definition ,a distance needs to satisfy certain properties.

The real numbers with the distance function d(x,y) = |y-x| given by the absolute difference ,and ,more generally, Euclidean n-space with the Euclidean distance , are complete metric spaces.The rational numbers with the same distance function also form a metric space ,but not a complete one.

The positive real numbers with distance function d(x,y) = | log (y/x)| is a complete metric space.Any normal vector space is a metric space by defining d(x,y) = ||y-x||, see also metrics on vector spaces .(If such a space is complete ,we call it a Banach space).

Examples:

- The Manhattan norm gives rise to the Manhattan distance, where the distance between any two points , or vectors, is the sum of the differences between corresponding coordinates.
- The cyclic Mannheim metric or Mannheim distance is a modulo variant the Manhattan metric.

The maximum norm gives rise to the Chebyshev distance or chessboard distance, the minimal number of moves a chess king would take to travel from x to y.

In 1992 B.C . Dhage proposed the notion of a D- metric space in an attempt to obtain analogous results to those for metric spaces ,but in a more general setting . In a subsequent series of papers Dhage presented topological structures in such spaces together with several fixed point results . These works have been the basis for a substantial number of results by other authors.Unfortunately, as we will show, most of the claims concerning the fundamental topological properties of D-metric spaces are incorrect , nullifying the validity of many results obtained in these spaces.

We begin by recalling the axioms of a D- metric space.

CHAPTER 1
METRIC SPACE : DEFINITION & EXAMPLES

1.1 METRIC SPACE

In mathematics, a metric space is a set together with a metric on the set. The matric space is a function that defines a concept of distance between any two members of the set, which are usually called points

1.1.1 Definition

A metric space is a pair (X, d) where X is a set and d is a mapping from $X \times X$ into R which satisfies the following conditions

D1) $d(x, y) \geq 0$, x, y ε X (non-negativity property)

D2) $d(x, y) = 0$, iff $x=y$ (zero property)

D3) $d(x, y) = d(y, x)$, x, y ε X (Symmetric property)

D4) $d(x, y) \leq d(x, z) + d(z, y)$, x, y, z ε X (Triangle property)

- The property D1 just states that a distance is always a non-negative numbers.
- The property D2 tells us that the distance identifies the points ie, if the distance between x and y is zero, it is because we are considering the same point.
- The property D3 states that a metric must measure distance symmetrically. i.e, it does not matter where we start measuring it.
- Finally the property D4, the triangular inequality is the generalization of the famous result that holds for the Euclidean distance in a plane.

A function satisfying the above four conditions is called a metric, and the structure (x, d) is called a metric space.

The function d is a non-negative real-valued function on X. the function d is also called distance function or simply distance.

1.1.2 Open ball

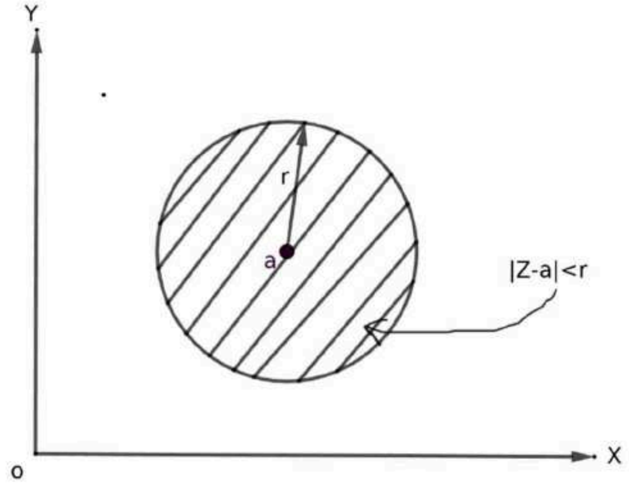
Let (X, d) be a metric space. If $a \in X$ and $r > 0$, then the set {x : x∈X, d(x,a) < r },

Denoted by $B_r(a)$, is called the open ball with centre a and radius r.

The open ball $B_r(a)$ on R is the bounded open interval (a – r, a + r) with mid-point a and total length 2r. The open ball $B_r(a)$ on C is the set

$\{x \in C : |z - a| < r\}$,

The open ball on C is also called the open disc.



1.1.3 Open Sets

Let (X, d) be a metric space. A set G ⊂ X is open if for each x∈G there is an r > 0 such that $B_r(x) \subset G$.

Examples :

(a) The set G = { z ϵ C : a < Re z < b }is open.

(b) The set S = { z ϵ C : Re z < 0} ∪ {0} is not open.

Note that the empty set ∅ and the full space X are open sets. Observe that in any metric space (X, d), each open ball is an open set.

1.1.4 Closed Sets

Let (X, d) be a metric space. The set G ⊂ X is said to be closed if the complement X – G is open.

1.1.5 Interior Points

Let (X, d) be a metric space and S a subset X. A point $x \in S$ is called an interior point of S if there exist an open ball $B_r(x)$ such that $B_r(x) \subset S$. The interior of S, denoted by S^0, is the set of all its interior points.

1.1.6 Closure points

Let (X, d) be a metric space and S a subset of X. A point $x \in X$ is called a closure point of S if every open ball centred on x contains at least one point of S. In other words, a point $x \in X$ is a closure point of S if

$$B_r(x) \cap S \neq \emptyset \text{ for all } r > 0.$$

The closure of S, denoted by $\bar{S}$, is the set of all its closure points

1.1.7 Closed Ball

Let (X, d) be a metric space. Let $a \in X$ and let $r > 0$. Then the set $\{x \in X : d(x, a) \leq r\}$ is called the closed ball with center a and radius r.

Here is an example. Let (X, d) be a discrete metric space (i.e. $d(x, y) = 1$ if $x \neq y$, $d(x, x) = 0$). Then

$$B_1(a) = \{x \in X : d(x, a) < 1\} = \{a\};$$

$$\{x \in X : d(x, a) \leq 1\} = X;$$

$$\overline{B_1(a)} = \{a\}.$$

1.1.8 Limit points

(X, d) be a metric space and S a subset of X. A point $x \in X$ is called a limit point (an accumulation point) of S if each open ball $B_r(x)$ contains atleast one point of S different form x. In other words, a point $x \in X$ is a limit of S if

$$B_r(x) \cap (S - \{x\}) \neq \emptyset \text{ for each } r > 0.$$

It is clear that every limit point of a set must be a closure point of that set.

The set of all limit points of S is called the derived set of S, and is denoted by S'.

Note that $\bar{S}$=S $\cup S'$. Also, note that S is closed if and only if it contains all its limit points.

Example : Let X = R and S = $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$.

0 is the limit of S.

1.1.9 Boundary

Let (X, d) be a metric space and S a subset of X. A point x∈X is called a boundary points of S if every open ball $B_r(x)$ intersects both S and S^c. In other words, a point x∈X is a boundary point of S if

$B_r(x) \cap S \neq \emptyset$ and $B_r(x) \cap S^c \neq \emptyset$ for all r > 0.

The boundary of S, denoted by δS, is the set of all its boundary points.

Note that $\delta S = \delta S^c$.

1.2 Convergence, Completeness

Just as a convergent sequence in real number can be thought of as a sequence of better and better approximations to a limit, so a sequence of "points" in a metric space can approximate a limit here. In a metric space X, every convergent sequence is a Cauchy sequence.

A metric space is said to be complete if every Cauchy sequence converges

1.2.1 Definition of convergence

Let (X, d) be a metric space. Let $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is said to be convergent to x in X if for every $\epsilon > 0$ there is an integer n_0 such that

$$d(x_n, x) < \epsilon \; for \; n \geq n_0.$$

In symbols, we write

$$lim \, x_n = x .$$

Note that a sequence in a metric space can have at most one limit.

1.2.2 Definition of complete metric space

A metric space (X, d) is complete if any of the following equivalent conditions are satisfied.

- Every Cauchy sequence of points in X has a limit that is also in X.
- Every Cauchy sequence in X converges in X (ie, to some point of X)

- The expansion constant of (X, d) is ≤ 2
- Every decreasing sequence of non-empty closed subsets of X, with diameters tending to 0, has a non-empty intersection.

1.2.3 Continuous Function

Let (X_1, d_1) and (X_2, d_2) be metric spaces. The function f : $X_1 \rightarrow X_2$ is said to be continuous at $a \in X_1$ if for each $\epsilon > 0$ there exists $\delta > 0$ such that,

$$d_2\big(f(x), f(a)\big) < \epsilon \text{ whenever } d_1(x, a) < \delta.$$

Note that δ depends on ϵ as well as on a.

The function f : $X_1 \rightarrow X_2$ is said to be continuous if it is continuous at each point of X_1.

1.2.4 Uniform continuity

Let (X_1, d_1) and (X_2, d_2) be metric spaces and let $f : X_1 \rightarrow X_2$ be a function. We say that f is uniformly continuous if for every $\epsilon > 0$ there exists $\delta > 0$ (depending only on ϵ) such that

$$d_2\big(f(x_1),\ f(x_2)\big) < \epsilon \text{ whenever } d_1(x_1, x_2) < \delta.$$

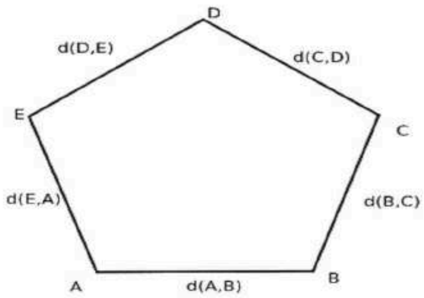
Note that every uniformly continuous function is continuous but the converse is not true. As an example, let $X_1 = (0,1]$ and $X_2 = R$ both with $(x,\ y) = |x - y|$. Then $f(x) = \frac{1}{x}$ is continuous but not uniformly continuous.

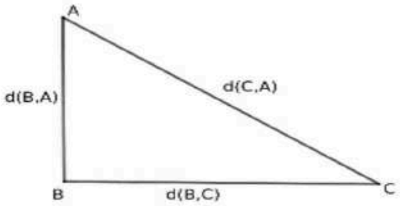
1.2.5 Notes

- If one defines d: X x X → R* U{0}, then the non-negativity property is redundant.
- The non-negativity property of a metric is a consequence of its other properties as for any x, y ε X

$$0 = d(x, x) \leq d(x, y) + d(y, x) = 2d(x, y)$$

- Once we are convinced about the underlying metric, we express (X, d) by mere X with the metric structure implied.In a metric space (X, d), $d(X_1, X_n) \leq \sum_{i=1}^{n-1} d(X_i, X_{i+1})$ for any x1,x2,x3, ...xn ε X. It is an extension of triangle inequality and known as polygonal inequality





- The conditions which d satisfies just mimic the properties of the distance we are accustomed for real numbers, and hence these properties bear same names as their real line counter part.

- Ignoring mathematical details, for any system of roads and terrains, the distance between two locations can be defined as the length of the shortest route connecting those locations. To be a metric there should not be any one way roads. The triangle inequality expresses the fact that detours are not shortcuts. If the distance between two points is zero, then the points are indistinguishable from one another.

1.3 EXAMPLES OF METRIC SPACES

1.3.1 Let $X=R^2$and d is defined as d: X x X $\rightarrow$R such that $d((x_1, x_2), (y_1, y_2))=$ $[(x_1-y_1)^2 + (x_2 - y_2)^2]^{1/2}$show that d is a metric space on R^2?

Solution.

$x = (x_1, x_2)$

$y = (y_1, y_2)$

$z = (z_1, z_2)$

D1) To show $d(x, y) \geq 0$

$D(x, y) = [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{½} \geq 0$

D2) To show $d(x, y)=0$ iff $x=y$

$d(x, y)=0 \Rightarrow [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{½} = 0$

$=> (x_1 - y_1)^2 + (x_2 - y_2)^2 = 0$

$=> (x_1 - y_1)^2 = 0$ & $(x_2 - y_2)^2 = 0$

$=> x_1 - y_1 = 0$ & $x_2 - y_2 = 0$

$=> x_1 = y_1$ & $x_2 = y_2$

$=> x = y$

D3) To show $d(x, y) = d(y, x)$

$d(x, y) = [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{½}$

$= [(y_1 - x_1)^2 + (y_2 - x_2)^2]^{½}$

$= d(y, x)$

D4) To show $d(x, y) \leq d(x, z) + d(z, y)$

Let $a_1 = x_1 - z_1$

$a_2 = x_2 - z_2$

$b_1 = z_1 - y_1$

$b_2 = z_2 - y_2$

$d(x, y) = [(a_1 + b_1)^2 + (a_2 + b_2)^2]^{½}$

$d(x, y) = [\sum_{k=1}^{2} (a_k + b_k)^2]^{½}$

$d(x, z) = [a_{12} + a_{22}]^{1/2}$

$=[\sum_{k=1}^{2} a_k^2]^{1/2}$

$D(z, y)= [b_1^2 + b_2^2]^{1/2}$

$=[\sum_{k=1}^{2} b_1^2]^{1/2}$

To show

$$[\sum_{k=1}^{2} a_k^2+b_k^2]^{1/2} \leq [\sum_{k=1}^{2} a_k^2]^{1/2} + [\sum_{k=1}^{2} b_k^2]^{1/2}$$

Squaring on both sides,

$$\sum_{k=1}^{2} a_k^2+b_k^2 \leq \sum_{k=1}^{2} a_k^2 + \sum_{k=1}^{2} b_k^2$$

$$\sum_{k=1}^{2} a_k b_k \leq (\sum_{k=1}^{2} a_k^2)^{1/2} (\sum_{k=1}^{2} b_k^2)^{1/2}$$

(which is Cauchy-schwartz inequality)

All 4 conditions are satisfied .

Therefore d is a metric on R^2

This is known as **Euclidean metric** on R^2.

1.3.2 Let X be a non-empty set and define a mapping d: $X \times X \to R$ as follows: $d(x, y) = \begin{pmatrix} 0, & when\ x=y \\ 1, & when\ x\neq y, \forall\ x,y\in X \end{pmatrix}$

Show that d is a metric on X.

Solution.

D1) $d(x, y) \geq 0$, by definition of d

D2) d(x, y) =0 iff x=y

D3) if x=y , then d(x, y)=0 =d(y, x) and

if x≠y, then d(x, y) =1= d(y, x).

Hence d(x, y)=d(y, x) for every x, y ϵ X

D4) Let x, y, z be any 3 elements in X.

If x=y, then d(x, y)=0, also d(x, z) ≥ 0 and d(z, y)=0.

Hence d(x, y) ≤ d(x, z) + d(z, y).

If x≠y, then either x≠y≠z or x≠y=z . ie, either d(x, y) = d(x, z) = d(z, y)=1 or d(x, y) = d(x, z)=1 and d(z, y) =0.

Hence in both situations, d(x, y) ≤ d(x, z) + d(z, y) Thus,

d(x, y) ≤ d(x, z) +d(z, y) for every x, y, z ϵ X

Hence, d is a metric on X and (X, d) is a metric space.

The metric space (X, d) so defined is known as **Discrete (Trivial) metric space**.

1.3.3 Let a mapping d: RxR →R be defined by d(x, y) = $\frac{|x-y|}{1+|x=y|}$ x, y ϵ R. Show that d is a metric on R.

Solution.

D1) d(x, y) ≥0, by definition of d.

D2) If d(x, y) =0

=> $\frac{|x-y|}{1+|x-y|}$ = 0,

=>|x-y| =0 => x-y =0 => x=y ie, d(x, y) =0 iff x=y

D3) d(x, y) = $\frac{|x-y|}{1+|x-y|}$=$\frac{|y-x|}{1+|y-x|}$ = d(y, x) x, y ϵ R

D4) d(x, y) = $\frac{|x-y|}{1+|x-y|}$ = $\frac{1-1+|x-y|}{1+|x-y|}$

= $\frac{1+|x-y|}{1+|x-y|}$ - $\frac{1}{1+|x-y|}$

$$=1-\frac{1}{1+|x-2+2-y|}$$

$$d(x, y) \leq 1-\frac{1}{1+|x-2|+|2-y|}=\frac{|x-2|+|2-y|}{1+|x-2|+|2-y|}$$

$$\Rightarrow d(x, y) \leq \frac{|x-2|}{1+|x-2|+|2-y|}+\frac{|2-y|}{1+|x-2|+|2-y|}$$

$$\Rightarrow d(x, y) \leq \frac{|x-2|}{1+|x-2|}+\frac{|2-y|}{1+|2-y|}=d(x,z)+d(z,y)$$

ie, $d(x, y) \leq d(x, z) + d(z, y)$

So d satisfies all the conditions. Therefore d is a metric on R.

1.3.4 Let R be the set of real numbers. Show that the function $d: R \times R \rightarrow R$ defined by $d(x, y) = |x-y|$ for every $x, y \in R$ is a metric on R.

Solution.

Here,

D1) $|x-y| \geq 0$, for every $x, y \in R \Rightarrow d(x, y) \geq 0$, $x, y \in R$

D2) $|x-y| = 0$ iff $x-y=0$ so that $d(x, y)=0$ iff $x=y$.

D3) $|x-y| = |y-x|$, for every $x, y \in R \Rightarrow d(x, y) = d(y, x)$, $x, y \in R$

D4) $|x-y| = |(x-z)+(z-y)|$

$\leq |x-z| + |z-y|$, for every $x, y, z \in R \Rightarrow d(x, y) \leq d(x, z) + d(z, y)$, $x, y, z \in R$.

Since d satisfies all conditions, d is a metric on R.

The above metric d is known as **usual metric on R** and the metric

space (R, d) is known as the **Real line**.

1.3.5 Let R be the set of real numbers. Show that the function $d: R \times R \rightarrow R$ defined by $d(x, y) + |x^2-y^2|$, for every $x, y \in R$ is a pseudometric on R which is not a metric on R.

Solution.

Here,

D1) $|x^2-y^2|\geq 0$, for every x, yϵR => d(x, y)≥o, ∀ x, yϵR.

D2) $d(x, x)=|x^2-x^2|=0$ ∀ xϵR

D3) $d(x, y)=|x^2-y^2| =|y^2-x^2| =d(y, x)$, ∀x, y$\epsilon$R

D4) $d(x, y) = |x^2-y^2|=|(x^2-z^2)+(z^2-y^2)| \leq |x^2-z^2|+|z^2-y^2|$

$\Rightarrow d(x, y) \leq d(x, z)+d(z, y)$, ∀ x, y, z$\epsilon$R

Hence d is a psedometric on R. We now show that d is not a metric on R.

Here, $d(x, y) = |x^2-y^2| =0 \leftrightarrow x^2-y^2=0$

$\leftrightarrow x^2=y^2$

$\leftrightarrow x=\pm y$

Showing that d(x, y)=0 does not always be x=y.

For example, we see that $d(1,-1) = |1^2-(-1)^2|=0$, where $1\neq -1$. Hence the function d is not a metric on R.

1.4 PRODUCT OF METRIC SPACES

1.4.1 Definition

Let (X_1, d_1) and (X_2, d_2) be two metric spaces. Then for any pair of points $x=(x_1, x_2)$ and $y=(y_1, y_2)$ in $X_1 x X_2$, consider the function $d:X_1 x X_2 \to R$ defined by:

$d(x, y)=\max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$ using the facts that d_1 and d_2 are metrics on X_1 and X_2 respectively. We shall prove that d is a metric on $X_1 x X_2$.

D1) $d(x, y) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}\geq 0$

D2) $d(x, y) = 0 \leftrightarrow \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} = 0$

$\leftrightarrow d_1(x_1, y_1) = 0$ and $d_2(x_2, y_2) = 0$

$\leftrightarrow x_1 = y_1$ and $x_2 = y_2$

$\leftrightarrow (x_1, x_2) = (y_1, y_2)$

$\leftrightarrow x = y$

D3) $d(x, y) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$

$= \max\{d_1(y_1, x_1), d_2(y_2, x_2)\}$

$= d(y, x)$ iv) For all $x = (x_1, x_2)$, $y = (y_1, y_2)$,

$z = (z_1, z_2)$ in $X_1 x X_2$, we have:

$d(x, z) + d(z, y) = \max\{d_1(x_1, z_1), d_2(x_2, z_2)\} + \max\{d_1(z_1, y_1),$

$d_2(z_2, y_2)\} \geq \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} = d(x, y)$

Thus $d(x, y) \leq d(x, z) + d(z, y)$ for every $x, y, z \epsilon X_1 x X_2$. Hence d is a metric on $X_1 x X_2$ and the metric space $(X_1 x X_2, d)$ is called the product of metric spaces (X_1, d_1) and (X_2, d_2).

CHAPTER 2

BOUNDED SETS AND EQUIVALENT METRICS

2.1 Bounded Sets in Metric Spaces

2.1.1 Definition

A subset S of a metric space (X,d) is bounded if there exist $x_0 \in X$ and $k \in R$ such that $d(x, x_0) \leq k$ for all $x \in S$.

If S satisfies the definition for some $x_0 \in X$ and $k \in R$, then it also satisfies the definition with x_0 replaced by any other point $x_1 \in X$ and k replaced by $k+d(x_0, x_1)$.

For if $d(x, x_0) \leq k$ then, $d(x, x_1) \leq d(x, x_0) + d(x_0, x_1) \leq k + d(x_0, x_1)$.

If S satisfies the definition then,

$$d(x, y) \leq d(x, x_0) + d(x_0, y) \leq 2k \text{ for all } x, y \in S.$$

2.1.2 Definition

If S is a non-empty bounded subset of a metric space with metric d then the diameter of S is $\sup\{d(x, y): x, y \in S\}$. The diameter of the empty set is 0.

If A is not bounded, then we take the diameter as infinity.

$$d(A)=\infty$$

From the above definition, a set has diameter zero if and only if the set is a singleton set and a nonempty set A is bounded if and only if it has a finite diameter.

2.1.3 Properties of bounded sets

(i) Any subset of a bounded set is bounded.

Let B be a bounded set with $d(x, y) \leq k$ for any $x, y \in B$. In particular this holds for x, y in any subset $A \subseteq B$. So A is bounded.

(ii) The union of a finite number of bounded sets is bounded.

Given a finite number of bounded sets $B_1, B_2, \ldots, B_n$ with diameter $k_1, k_2, \ldots k_N$ respectively.

Let $k = \max_n k_n$

Pick a representative point from each set $a_n \in B_n$ and take the maximum distance between any two, $\check{k} = \max_{m,n} d(a_m, a_n)$, it certainly exists as there are only a finite number of such pairs.

Now for any two points x, y $\in \cup B_n$, that is $x \in B_i$, $y \in B_j$ and using the triangle inequality twice,

$$d(x, y) \leq d(x, a_i) + d(a_i, a_j) + d(a_j, y)$$

$$\leq ki + \check{k} + kj$$

$$\leq 2k + \check{k}$$

An upper bound for the distances between points is $\bigcup_{n=1}^{N} B_n$ is found.

2.1.4 Examples

(i) In any metric space, finite subsets are basis. In N, only the finite subsets are bounded (since $d(a_0, a_n) \leq N$ for all n implies $n \leq N$). Consequently, N, Q, R and C are all unbounded.

(ii) In a discrete metric space, every subset is bounded. A metric space may be "large" (non - separable) yet be bounded.

(iii) A set B is bounded ⇔it is a subset of a ball, There exists r > 0, a $_x$, $B \subseteq B_r(a)$

Proof

Balls (and their subsets) are obviously bounded.

For every x, y $\in B_r(a)$,

$$d(x, y) \leq d(x, a) + d(y, a) < 2r$$

Conversely if a non - empty set is bounded by R > 0, pick any points $a \in X$ and $b \in B$ to conclude $x \in B_r(a)$:

For every $x \in B$, $d(x, a) \leq d(x, b) + d(b, a)$

$< R + 1 + d(b, a) = r$

(iii) The set $[0, 1] \cup [2, 3] \subseteq R$ is bounded because it is the union of two bounded sets.

2.1.5 Theorem

Every Cauchy sequence in a metric space (X, d) is bounded. That is, the different terms of a Cauchy sequence form a bounded set.

Proof

If (x_n) is a Cauchy sequence in (X, d) taking $\varepsilon = 1$, there exists n_0 such that

$d(x_n, x_0) \leq 1$ for all $n \geq n_0$

For other values of n from 1 to n_0, let us take

$M = \max d(x_n, x_0) < \infty$, $1 \leq n \leq n_0$

Hence, we have $d(x_n, x_0) \leq M + 1$ for all n.

So for all m, n we have $d(x_n, x_m) \leq d(x_n, x_{n0}) + d(x_m, x_{n0})$

$\leq 2(M + 1)$

Hence, $d(x_n, x_m) \leq 2(M + 1) < \infty$.

Therefore, every Cauchy sequence in a metric space is bounded.

2.2 Totally Bounded Set

2.2.1 Definition

A subset $B \subseteq X$ is totally bounded when it can be covered by a finite number of

ε-balls, however small their radii ε,

For every $\varepsilon > 0$, there exists $N \in N$, there exists $a_1, a_2, \ldots, a_N \in X$

$B \in \bigcup_{n=1}^{N} B_\varepsilon(a_o)$

2.2.2 Easy consequences

(i) Any subset of a totally bounded set is totally bounded (the same ε-cover of the parent covers the subset).

(ii) A finite union of totally bounded sets is totally bounded (the finite collection of ε-covers remains finite).

(iii) A totally bounded set is bounded (it is a subset of a finite number of bounded balls).

2.2.3 Examples

(i) The interval [0, 1] is totally bounded in R because it can be covered by the balls. $B_\varepsilon(n\varepsilon)$ for n = 0, 1,…,N where $\frac{1}{\epsilon} - 1 < N \leq w\ \frac{1}{\epsilon}$

(ii) Not all bounded sets are totally bounded. For example, in a discrete metric space, any subset is bounded but only finite subsets are totally bounded (take ε < 1).

2.3 Equivalent Metrics

2.3.1 Definition

Let X be a set and let d and d` be two metrics on X. We say that d and d` are equivalent if for every subset U C X,

U is open in (X, d) ⇔ U is open in (X, d`)

In otherwise, two metrics are equivalent if they define the same collections of open subsets on X. It is denoted as d ~ d`.

2.3.2 Examples

1. Let (X, d) be a metric space and ρ be a function on X × X defined by,

$$\rho(x, y) = \min\{1, d(x, y)\} \text{ for every } x, y \in X \text{ then,}$$

(a) (x, ρ) is a bounded metric space

(b) ρ is equivalent to d

Solution

(a) Given that $\rho (x, y) = \min \{1, d(x, y)\}$ -------------(1)

(i) $d(x, y) \geq 0 \rightarrow \min \{1, d(x, y)\} \geq 0 \rightarrow \rho (x, y) \geq 0$

(ii) $\rho(x, y) = 0 \rightarrow \min \{1, d(x, y)\} = 0 \rightarrow d(x. y) = 0 \rightarrow x = y$

(iii)$\rho(x, y) = \min \{1, d(x, y)\} = \min \{1, d(y, x)\} = \rho(y, x)$, for every $x, y \in X$.

(iv) Let x, y, z be any 3 points of X. If atleast one of $d(x, y)$ and $d(y, z)$ is ≥ 1 also $d(x, z) \geq 1$

Then $\min \{1, d(x, y)\} = 1$ so that $\rho(x, y) = 1$,by (1)

Therefore, $\rho(x, y) + \rho(y, z) \geq 1 \geq \rho(x, z)$.

Also in case when $d(x, y) < 1$ and $d(y, z) < 1$,

we have $\rho(x, y) = \min \{d(x, y), 1\} = d(x, y)$ and

$\rho(y, z) = \min \{d(y, z), 1\} = d(y, z)$

Therefore $\rho(x, y) + \rho(y, z) = d(x, y) + d(y, z)$

$\geq d(x, z)$, by triangle inequality

$\geq \rho(x, z)$, since $\rho(x, z) \leq d(x, z)$,by (1)

Thus $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$, for every $x, y \in X$.

Hence ρ is a metric on X and (X, ρ) is a metric space.

From (1), $\rho(x, y) \leq 1$, for every $x, y \in X$

Hence $d(X, \rho)$ is a bounded metric space.

(b) To show that ρ is equivalent to d.

For this purpose we must show that every open set is (X, ρ) is open in (X, d) and vice versa.

Let G be any open subset of X in (X, ρ) and let x be any point of G.
Then there exist an open set

$\{y \in X : \rho(x, y) < r\} \subseteq G$ ------- (2)

From (1), $\rho(x, y) \leq d(x, y)$ for every $x, y \in X$

Hence using (2) we get

$\{y \in X : d(x, y) < r\} \subseteq G \subseteq \{y \in X : \rho(x, y) < r\} \subseteq G$

showing that each point of G is the center of an open sphere in (X, d) contained in G. Hence G is also open in (X, d) and therefore, every open set in (X, ρ) is open in (X, d).

Next, let H be an open set in (X, d).

Then for each $x \in H$, there exists an open sphere

$\{y \in X : d(x, y) , r\} \subseteq H$

$r` = \min \{1, r\}$ so that $r` \leq r$ Then

we have

$\{y \in X : \rho(x, y) < r`\} \subseteq \{y \in X : d(x, y) < r\} \subseteq H$

showing that each point of H is the center of an open sphere in (X, ρ) contained in H.

Hence H is also open in (X, ρ).

Therefore, every open set in (X, d) is open in (X, ρ).

Hence d and ρ are equivalent metrics.

2. Let (X, d) be a metric space and let

$$d^*(x, y) = \frac{d(x,y)}{1 + d(x,y)} \quad , \text{for all } x, y \in X$$

prove that d* is a bounded metric on X, which is equivalent to d.

Solution:

Given $d^*(x, y) = \dfrac{d(x,y)}{1 + d(x,y)}$, for all $x, y \in X$ ---------(1)

(i) Since $d(x, y) \geq 0$, (1) show that $d^*(x, y) \geq 0$

(ii) $d^*(x, y) \geq 0 \Leftrightarrow d(x, y) = 0$, by (1)

$\Leftrightarrow x = y$, since d is metric on X.

(iii) $d^*(x, y) = \frac{d(x,y)}{1 + d(x,y)} = \frac{d(y,z)}{1 + d(y,z)} = d^*(y, x)$

since d is a metric on X ➔ $d(x, y) = d(y, x)$.

(iv) For all x, y, z in X, we have

$$d^*(x, y) + d^*(y, z) = \frac{d(x,y)}{1 + d(x,y)} + \frac{d(y,z)}{1 + d(y,z)}$$

$$\geq \frac{d(x,y)}{1 + d(x,y) + d(y,z)} + \frac{d(y,z)}{1 + d(x,y) + d(y,z)}$$

since d is metric ➔ $d(x, y) \geq 0$ and $d(y, z) \geq 0$

Lemma

Let d_1, d_2 and d_∞ be the following metrics on R^2:

- $d_1((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$
- $d_2((x_1, y_1), (x_2, y_2)) = \sqrt{((x_1 - x_2)^2 + (y_1 - y_2)^2)}$
- $d_\infty((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$

Then for each $(x_1, y_1), (x_2, y_2) \in R^2$,

$$\frac{1}{2}[d_1((x_1, y_1), (x_2, y_2))] \leq \frac{1}{\sqrt{2}}[d_2((x_1, y_1), (x_2, y_2))]$$

$$\leq d_\infty((x_1, y_1), (x_1, y_1))$$

Proof

By definition of the metric d_2,

$$d_2((x_1, y_1), (x_2, y_2)) = \sqrt{((x_1 - x_2)^2 + (y_1 - y_2)^2)}$$

$$\leq \sqrt{(\max\{|x_1 - x_2|^2, |y_1 - y_2|^2\} + \max\{|x_1 - x_2|^2, |y_1 - y_2|^2\}}$$

$$= \sqrt{(2\max\{|x_1 - x_2|^2, |y_1 - y_2|^2\})}$$

$$= \sqrt{2}\, d_\infty((x_1, y_1), (x_2, y_2))$$

That proves the inequality between d_2 and d_∞.

Now we compare d_1 and d_2.

Since everything is positive, proving the inequality between d_1 and d_2 in the statement is equivalent to proving the resulting inequality after squaring both sides. In other words, we have to prove

$$\frac{1}{4}[\,|x_1 - x_2| + |y_1 - y_2|]^2 \le \frac{1}{2}[(x_1 - x_2)^2 + (y_1 - y_2)^2]$$

Let $a = |x_1 - x_2|$ and $b = |y_1 - y_2|$

Then above inequality is same as

$(a + b)^2 \le 2(a^2 + b^2)$ which is equivalent to $a^2 + b^2 \le a^2 + b^2 + (a - b)^2$ and this last inequality is obviously true.

Therefore, the inequality between d_1 and d_2 in the statement is true.

CHAPTER 3

D-METRIC SPACES

3.1 Definition

A real function D on $X \times X \times X$ is said to be a D-metric on X if

D(1) $D(x, y, z) \geq 0$ for all $x, y, z \in X$(non-negativity)

D(2) $D(x, y, z)=0$ if and only if $x=y=z$ (coincidence)

D(3) $D(x, y, z)=D(p(x, y, z))$ for every $x, y, z \in X$ and for any permutation $p(x, y, z)$ of x, y, z (symmetry)

D(4) $D(x, y, z) \leq D(x, y, u)+ D(x, u, z)+D(u, y, z)$ for every $x, y, z, u \in X$ (tetrahedral inequality)

A D-metric space is a pair (X, D) ,where D is a D-metric on X.

3.2 Examples of D metric spaces

3. 2.1 Example

For $x, y, z \in R$ define

$D_1(x, y, z)= |x-y| + |y-z| + |z-x|$

$D_\infty(x, y, z)=\max\{|x-y| , |y-z| , |z-x| \}$

Then (R,D_1) and (R,D_∞) are D-metric spaces.

3 .2.2 Example

Define a function D on $X \times X \times X$ by

$$D(x, y, z) =\begin{cases} 0 & if\ x = y = z \\ 1 & otherwise \end{cases}$$

then D is a D- metric on X and is called the discrete D-metric on X

3.3 Notes

3.3.1 Remark

The D-metrics given in examples1.2,1.3 satisfy the following properties:

For every x, y, z, u, v in X

D(5) $D(x, y, z) \leq D(x, z, z) + D(z, y, y)$

D(6) $D(x, x, y) = D(x, y, y)$

D(7) $D(x, y, y) \leq D(x, y, z)$

D(8) $D(x, y, z) \leq D(x, u, v) + D(u, y, v) + D(u, v, z)$

3.3.2 Remark

Clearly D(7)⇒D(6).The following example shows that D(6) does not necessarily imply D(7)

Suppose X has atleast three elements. Define D on $X \times X \times X$ by

$$D(x, y, z) = \begin{cases} 0 & if\ x = y = z \\ \frac{1}{2} & if\ x,\ y,\ z\ are\ distinct \\ 1 & otherwise \end{cases}$$

Then (X, D) is a D=metric space in which D(6) holds but D(7) fails to hold

3.4 Some theorems of D-metric spaces

3.4.1 Theorem

If in a D-metric space (X, D), D(5) holds then each of the function $d: X \times X \to R^+$ defined by

(1) for $1 \leq p < \infty$ $d(x, y) = \{D^p(x, y, y) + D^p(x, x, y)\}^{1/p}$ is a metric on X.

(2) $d(x,y) = \max\{D(x, y, y), D(x, x, y)\}$ for all $x, y \in X$ is a metric on X.

Proof

we prove (1) ,The proof of (2) is similar.

Clearly $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$

Let $x, y, z \in X$.

For $1 \leq p < \infty$ $d(x, y) = \{D^P(x, y, y) + D^p(x, x, y)\}^{1/p}$

$= \{D^P(y, y, x) + D^P(y, x, x)\}^{1/p}$

$= \{D^P(y, x, x) + D^p(y, y, x)\}^{1/p}$

$=d(y, x)$

$d(x,y)= \{D^P(x, y, y)+ D^P(x, x, y)\}^{1/p}$

$\leq\{(D(x, z, z)+D(z, y, y))^p + (D(x, x, z)+(D(z, z, y))^p\}^{1/p}$ (by D(5))

$\leq\{D^P(x, z, z) +D^p(x, x, z)\}^{1/p} + \{D(z, y, y)+D^p(z, z, y)\}^{1/p}$

$=d(x, z) + d(z, y)$

Hence d is a metric on X.

3.4.2 Theorem

Let D be a real function on $X \times X \times X$ satisfying D(1),D(2),D(3),D(7) and D(8) then D is a D-metric on .

Proof : It is enough to prove D($).Let x, y, z $\in$ X

$D(x, y, z) \leq D(x, u, u) + D(u, y, u) + D(u, u, z)$ (using D(8))

$\leq D(x, y, u) + D(x, u, z) + D(u, y, z)$ for every $u \in X$ (by D(7))

Thus D(4) holds and hence D is a D-metric on X.

3.4.3 Theorem

Let (X, d) be a metric space. Define real functions D_1,D_∞,D_3,D_4 on $X \times X \times X$ by

$D_1(x, y, z) = d(x, y) + d(y, z) + d(z, x), D_\infty(x, y, z) = max\{d(x, y), d(y, z), d(z, x)\}$,

$D_3(x, y, z) = \{D_1(x, y, z) if\ x, y, z\ are\ distinct\ D_\infty(x, y, z) otherwise$

$D_4(x, y, z) =\{D_\infty(x, y, z) if\ x, y, z\ are\ distinct\ \ D_1(x, y, z) otherwise$,

Then D_1, D_∞ ,D_3 ,D_4 are D-metrics on X.

proof

That D_1 , D_∞ are D-metrics can be proved *easily. As* the proofs for D_3 , D_4 are similar, we verify

that D_4 is D-metric.

For *this, it* is enough to verify the tetrahedral inequality.

$Let\ x, y, z, u \in X$.

Case(i): x, y, z are distinct.

Without loss of generality , we assume that $d(x, y) \leq d(y, z) \leq d(z, x)$.

Subcase(i):

if u $\notin \{x, y, z\}$ $then\ D_4(x, y, z) = d(x, z) \leq d(x, u) + d(u, z) \leq D_4(x, y, u) + D_4(x, u, z) + D_4(u, y, z)$

Subcase(ii):

if u = x then $D_4(x, y, z) = D_4(u, y, z) \leq D_4(x, y, u) + D_4(x, u, z) + D_4(u, y, z)$

the proof is similar if u = =y or u = z.

Case(ii):Assume x = y ≠ z.

Subcase(i):

if u $\notin$ $\{y, z\}$ then $D_4(x, y, z) = d(y, z) + d(z, y) \leq d(y, u) + d(u, z) + d(x, u) = d(u, y) \leq D_4(x, y, u) + D_4(x, u, z) + D_4(u, y, z)$

Subcase(ii):

if u = y then $D_4(x, y, z) = D_4(x, u, z) \leq D_4(x, y, u) + D_4(x, u, z) + D_4(u, y, z)$

Subcase(ii):

if u = z ≠ y $then\ D_4(x, y, z) = D_4(x, y, u) \leq D_4(x, y, u) + D_4(x, u, z) + D_4(u, y, z)$. Hence D_4 is a D-metric on X.

In this section we discuss various types of convergence associated with a D-metric.

3.5 Definitions

3 .5. 1 Definition

A sequence $\{ x_n \}$ in a D-metric space (X, D) is said to be D-convergent (or convergent) if there exists an element x in X such that for given $\varepsilon > 0$, there exists a positive integer m_0 such that $D(x_n, x_m, x) < \varepsilon$ for all $m \geq m_0$, $n \geq m_0$

In such a case, we say that $\{x_n\}$ converges to x and x is a limit of $\{x_n\}$.

3. 5 .2 Definition

A sequence $\{x_n\}$ in a D-metric space (X, D) is said to be D-Cauchy (or Cauchy) if for given $\varepsilon > 0$, there exists a positive integer m_0 such that $D(x_n, x_m, x_p) < \varepsilon$ for all $m \geq m_0$, $n \geq m_0$, $p \geq m_0$.

3.5.3 Definition

Let (X, D) be a D-metric space and $\{x_n\}$ be a sequence in X, we say that $\{x_n\}$ converges strongly to an element x in X if

(i) $D(x_n, x_m, x) \to 0$ as $m,n \to \infty$

(ii) $\{D(y, y, x_n)\}$ converges to $D(y, y, x)$ for all $y \in X$.

3.5.4 Definition

Let (X,D) be a D-metric space and $\{x_n\}$ be a sequence in X , we say that $\{x_n\}$ converges very strongly to an element x in X if

(i)$D(x_n,x_m,x) \to 0$ as $m,n \to \infty$

(ii) $\{D(y,z, x_n)\}$ converges to $D(y,z, x)$ for all $y,z \to X$.

3.5.5 Remark

It is clear that very strong convergence implies convergence, but converse is not true. It is also clear that in a D-metric space, every strongly convergent sequence has a unique strong limit where as limits are not unique under convergence. We now prove that convergence and strong convergence are equivalent in certain cases.

3.6 Theorem.

Let (X,D) be a D-metric space satisfying D(5). Then $\{x_n\}$ converges to x in (X,D) strongly if and only if $\{x_n\}$ converges to x in (X,D) and $D(x, x, x_n) = 0$.

Proof. Let $\{x_n\}$ be a D-convergent sequence in X with limit x , $D(x, x, x_n) = 0$ and $\varepsilon > 0$. Then there exists a positive integer m_0 such that $D(x,x,x_n) < \varepsilon$ for all $n \geq m_0$. Let $y \in X$.

For $n \geq m_0$ $\quad D(y,y,x_n) \leq D(x_n,x,x) + D(x,y,y,)$ (By D(5))

this implies that $|D(y,y,x_n) - D(y,y,x)| \leq D(x,x,x_n) < \varepsilon$ for all $n \geq m_0$.

3.7 Theorem

Let (X,D) be a D-metric space satisfying D(7),D(8) then the real function d on $X \times X$ defined by d(x, y)=D(x, y, y) is a metric on X and the following are equivalent.

(1) $\lim_{n\to\infty} x_n = x$ in (X, d)

(2) $\lim_{n\to\infty} x_n = x$ in (X, D)

(3) $\lim_{n\to\infty} x_n = x$ strongly in (X, D)

Proof: From proposition it is clear that d is a metric on X.

Assume (1) $\lim_{n\to\infty} x_n = x$ in (X, d)

Let $\varepsilon > 0$ .Then there exists a positive integer m_0 such that $d(x, x_n) < \varepsilon/2$ for all $n \geq m_0$.

For $n, m \geq m_0$, $D(x, x_n, x_m) \leq D(x, x, x_m) + D(x, x, x_n)$

$= d(x, x_m) + d(x, x_n) < \varepsilon$ (using D(8)).Thus (1) ⇒(2).

Assume (2) $\lim_{n\to\infty} x_n = x$ in (X, D)

Let $\varepsilon > 0$ .Then there exists a positive integer m_0 such that $D(x_n, x_m, x) < \varepsilon$ for all $m \geq m_0$, $n \geq m_0$.

For $y \in X$ and $n \geq m_0$, $D(y, y, x_n) \leq D(x, y, x_n)$ (by D(7))

$\leq D(x, x, x_n) + D(y, x, x)$ (by D(8))

$= D(x, x_n, x_n) + D(y, y, x)$

(by D(6),since D(7) implies D(6))

This implies that $| D(y, y, x_n) - D(y, y, x) | \leq D(x, x_n, x_n) < \varepsilon/2$ for all $n \geq m_0$.

Hence $\{D(y, y, x_n)\}$ converges to D(y, y, x) for all y in X.

Thus (2)⇒(3) , (3)⇒(2) is trival ,

Assume (2) $\lim_{n\to\infty} x_n = x$ in (X, D)

Let $\varepsilon > 0$ .Then there exists a positive integer m_0 such that $D(x_n, x_m, x) < \varepsilon$ for all $m \geq m_0$, $n \geq m_0$

For , $n \geq m_0$, $d(x, x_n) = D(x, x, x_n) \leq D(x, x_m, x_n) < \varepsilon$. (by D(7))

Hence $\lim_{n\to\infty} x_n = x$ in (X, d). Thus (2) ⇒ (1).

CONCLUSION

In this project I tried to study the distance between points in a set. In everyday life, the question “ what is the distance between A and B ?” is ambiguous. The standard definition of distance is the length of the straight line segment with one end at A and the other end at B . It generalizes the idea of distance between two points on the real line.

The idea of distance between two points on real line plays a vital role in formulating the basic thing like limit, continuity, differentiability, convergence in Real Analysis.By doing this project, I studied about different metric spaces such as discrete, dilation, usual, Euclidean etc. I heard that metric space is very interesting topic. So I decided to take it as my project topic.

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