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Project Work

Entitled

"Fibonacci Numbers"

Submitted to

DEPARMENT OF MATHEMATICS

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The Award of B.Sc. Degree in Mathematics

By

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CERTIFICATE

This is to certify that **Miss.** Manisha Madhukar Khambe of the class B.Sc. III has satisfactorily completed the project work on the title *"Fibonacci Numbers"* as a partial fulfillment of the practical course for the award of the B.Sc. Degree in Mathematics by Shivaji University, Kolhapur.

Place: Kolhapur

Date:

Project Guide | Examiner | Head of Department

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DECLARATION

It is hereby declare that the work reported in the project entitled *"Fibonacci Numbers"* is completed and written by me and has not been copied from anywhere.

Place: Kolhapur

Date:

ACKNOWLEDGEMENT

I wish to express my deep sense of gratitude towards Mr. S.P. Patankar , Mr. S.T.Sutar, Department Of Mathematics, Vivekanand College, and Kolhapur for their valuable guidance to complete this project within time.

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My sincere thanks to all those who have directly or indirectly involved in this project work.

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Introduction

The term \Fibonacci numbers" is used to describe the series of numbers generated by the pattern

1; 1; 2; 3; 5; 8; 13; 21; 34; 55; 89; 144......

where each number in the sequence is given by the sum of the previous two terms.

This pattern is given by $u_1 = 1, u_2 = 1$and the recursive formula

$$u_n = u_{n-1} + u_{n-2},\ n > 2$$

First derived from the famous \rabbit problem" of 1228, the Fibonacci numbers were originally used to represent the number of pairs of rabbits born of one pair in a certain population. Let us assume that a pair of rabbits is introduced into a certain place in the first month of the year. This pair of rabbits will produce one pair of offspring every month, and every pair of rabbits will begin to reproduce exactly two months after being born. No rabbit ever dies, and every pair of rabbits will reproduce perfectly on schedule. So, in the first month, we have only the first pair of rabbits. Likewise, in the second month, we again have only our initial pair of rabbits. However, by the third month, the pair will give birth to another pair of rabbits, and there will now be two pairs. Continuing on, we find that in month four we will have 3 pairs, then 5 pairs in month four, then 8,13,21,34,...,etc, continuing in this manner. It is quite apparent that this sequence directly corresponds with the Fibonacci sequence introduced above, and indeed, this is the first problem ever associated with the now-famous numbers.

Now that we have seen one application of the Fibonacci numbers and established a basic definition.

Simple Properties of the Fibonacci Numbers

Lemma 1. Sum of the Fibonacci Numbers

The sum of the first 'n' Fibonacci numbers can be expressed as

$$u_1 + u_2 + \cdots + u_{n-1} + u_n = u_{n+2} - 1$$

Proof. From the defination of the Fibonacci sequence, we know

$$u_1 = u_3 - u_2,$$

$$u_2 = u_4 - u_3,$$

$$u_3 = u_5 - u_4,$$

.

.

$$u_{n-1} = u_{n+1} - u_{n+2},$$

$$u_n = u_{n+2} - u_{n+1}.$$

We now add these equations to find

$$u_1 + u_2 + \cdots + u_{n-1} + u_n = u_{n+2} - u_2$$

Recalling that $u_2 = 1$ we see this equation is equivalent to our initial conjecture of

$$u_1 + u_2 + \cdots + u_{n-1} + u_n = u_{n+2} - 1.$$

Lemma 2. Sum of Odd Terms

The sum of the odd terms of the Fibonacci sequence

$$u_1 + u_3 + u_5 + \cdots + u_{2n-1} = u_{2n}.$$

Proof. Again looking at individual terms, we see from the defination of the sequence that

$$u_1 = u_2,$$

$$u_3 = u_4 - u_2,$$

$$u_5 = u_6 - u_4,$$

$$u_{2n-1} = u_{2n} - u_{2n-2}$$

If we now add these equations term by term, we are left with the required result from above.

Lemma 3. Sum of Even Terms

The sum of the even terms of the Fibonacci sequence

$$u_2 + u_4 + u_6 + \cdots + u_{2n} = u_{2n+1} - 1.$$

Proof: From lemma 1, we have

$$u_1 + u_2 + \cdots + u_{n-1} + u_{2n} = u_{2n+2} - 1$$

Subtracting our equation for the sum of odd terms, we obtain

$$u_2 + u_4 + \cdots + u_{2n} = u_{2n+2} - 1 - u_{2n} = u_{2n+1} - 1.$$

as we desired.

Lemma 4. Sum of Fibonacci Numbers with Alternating Signs

The sum of the Fibonacci numbers with alternating signs

$$u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n+1} u_n = (-1)^{n+1} u_{n-1} + 1$$

Proof: Building further from our progress with sums, we can subtract our even sum equation from our odd sum equation to

(1) $$u_1 - u_2 + u_3 - u_4 + \cdots + u_{2n-1} - u_{2n} = -u_{2n-1} + 1$$

Now, adding u2n+1 to both sides of this equation, we obtain

$$u_1 - u_2 + u_3 - u_4 + \cdots - u_{2n} + u_{2n+1} = u_{2n+1} - u_{2n-1} + 1,$$

or

(2) $$u_1 - u_2 + u_3 - u_4 + \cdots - u_{2n} + u_{2n+1} = u_{2n} + 1$$

Combining equations (1) and (2), we arrive at the sum of Fibonacci numbers with alternating signs:

$$u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n+1} u_n = (-1)^{n+1} u_{n-1} + 1$$

Thus far, we have added the individual terms of simple equations to derive lemmas regarding the sums of Fibonacci numbers. We will now use a similar technique

To find the formula for the sum of the squares of the first 'n' Fibonacci numbers.

Lemma 5: Sum of Squares

The sum of the squares of the first 'n' Fibonacci numbers

$$u_1^2 + u_2^2 + \cdots + u_{n-1}^2 + u_n^2 = u_n u_{n+1}.$$

Proof: Note that

$$u_k u_{k+1} - u_{k-1} u_k = u_k(u_{k+1} - u_{k-1}) = u_k^2.$$

If we add the equation

$$u_1^2 = u_1 u_2,$$

$$u_2^2 = u_2 u_3 - u_1 u_2,$$

$$u_3^2 = u_3 u_4 - u_2 u_3,$$

$$u_n^2 = u_{n-1} u_n$$

term by term, we arrive at the formula we desired.

Until now, we have primarily been using term-by-term addition to find formulas for the sums of Fibonacci numbers. We will now use the method of induction to prove the following important formula.

Lemma 6: Another Important Formula

$$u_{n+m} = u_{n-1} u_m + u_n u_{m+1}.$$

Proof: We will now begin this proof by induction on m. For m = 1,

$$u_{n+1} = u_{n-1} + u_n$$

$$= u_{n-1} u_1 + u_n u_2,$$

which we can see holds true to the formula. The equation for m = 2 also proves true for our formula, as

$$u_{n+2} = u_{n+1} + u_n$$

$$= u_{n-1} + u_n + u_n$$

$$= u_{n-1} + 2u_n$$

$$= u_{n-1}u_2 + u_n u_3.$$

Thus, we have now proved the basis of our induction. Now suppose our formula to be true for m = k and for m = k + 1. We shall prove that it also holds for m = k + 2.

So, by induction, assume

$$u_{n+k} = u_{n-1}u_k + u_n u_{k+1}$$

and

$$u_{n+k+1} = u_{n-1}u_{k+1} + u_n u_{k+3}$$

If we add these two equations term by term, we obtain

$$u_{n+k} + u_{n+k+1} = u_{n-1}(u_k + u_{k+1}) + u_n(u_{k+1} + u_{k+2})$$

$$u_{n+k+2} = u_{n-1}u_{k+2} + u_n u_{k+3},$$

which was the required result. So, by induction we have proven our initial formula holds true for m = k + 2, and thus for all values of m. _

Lemma 7: Difference of Squares of Fibonacci Numbers

$$u_{2n} = u_{n+1}^2 - u_{n-1}^2$$

Proof. Continuing from the previous formula in Lemma 7, let m = n. We obtain

$$u_{2n} = u_{n-1}u_n + u_n u_{n+1},$$

or

$$u_{2n} = u_n(u_{n-1} + u_{n+1}).$$

Since

$$u_n = u_{n+1} - u_{n-1},$$

we can now rewrite the formula as follows:

$$u_{2n} = (u_{n+1} - u_{n-1})(u_{n+1} + u_{n-1}),$$

or

$$u_{2n} = u_{n+1}^2 - u_{n-1}^2$$

Thus, we can conclude that for two Fibonacci numbers whose positions in the sequence difference by two, the difference of squares will again be a Fibonacci number.

Fibonacci Numbers and Pascal's Triangle: The Fibonacci numbers share an interesting connection with the triangle of binomial coefficients known as Pascal's triangle.

Pascal's triangle typically takes the form:

(3)

```
  1
 1 1
 1 2 1
1 3 3 1
1 4 6 4 1
```

In this depiction we have oriented the triangle to the left for ease of use in our future application. Pascal's triangle, as may already be apparent, is a triangle in which the topmost entry is 1 and each following entry is equivalent to the term directly above plus the term above and to the left.

Another representation of Pascal's triangle takes the form:

(4)

$$\begin{matrix} C_0^0 \\ C_1^0 & C_1^1 \\ C_1^0 & C_2^1 & C_2^2 \\ C_3^0 & C_3^1 & C_3^2 & C_3^3 \\ C_4^0 & C_4^1 & C_4^2 & C_4^3 & C_4^4 \end{matrix}$$

In this version of Pascal's triangle, we have $C_j^i = \frac{k!}{i!(k-i)!,}$, where i represents the column and k represents the row the given term is in. Obviously, we have designated the first row as row 0 and the first column as column 0.

Finally, we will now depict Pascal's triangle with its rising diagonals.

1						
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1

Figure 1: Pascal's Triangle with Rising Diagonals

The diagonal lines drawn through the numbers of this triangle are called the "rising diagonals" of Pascal's triangle. So, for example, the lines passing through 1; 3; 1 or 1; 4; 3 would both indicate different rising diagonals of the triangle. We now go on to relate the rising diagonals to the Fibonacci numbers

Theorem 1: The sum of the numbers along a rising diagonal in Pascal's triangle is a Fibonacci number.

Proof: Notice that the topmost rising diagonal only consists of 1, as does the second rising diagonal. These two rows obviously correspond to the first two numbers of the Fibonacci sequence.

To prove the proposition, we need simply to show that the sum of all numbers in the $(n-2)^{nd}$diagonal and the $(n-1)^{st}$diagonal will be equal to the sum of all numbers in the nth diagonal in Pascal's triangle.

The $(n-2)^{nd}$ diagonal consists of the number

$$C_{n-3}^{0}, C_{n-4}^{1}, C_{n-5}^{2}, \dots$$

and the $(n-1)^{st}$ diagonal has the numbers

$$C_{n-2}^{0}, C_{n-3}^{1}, C_{n-4}^{2}, \dots$$

We can add these numbers to find the sum

$$C_{n-2}^{0} + (C_{n-3}^{0} + C_{n-3}^{1}) + (C_{n-4}^{1} + C_{n-4}^{2}) + \cdots$$

However, for the binomial coefficients of Pascal's triangle,

$$C_{n-2}^{0} = C_{n-1}^{0} = 1$$

and

$$C_{k}^{i} + C_{k}^{i+1} = \frac{k(k-1)...(k-i+1)}{1.2...i} + \frac{k(k-1)...(k-i+1)(k-i)}{1.2...i.(i+1)}$$

$$= \frac{k(k-1)...(k-i+1)}{1.2...i}(1 + \frac{k-i}{i+1})$$

$$= \frac{k(k-1)...(k-i+1)}{1.2...i}\,\frac{i+1+k-1}{i+1}$$

$$= \frac{(k+1)k(k-1)...(k-i+1)}{1.2...i.(i+1)}$$

$$= C_{k+1}^{i+1}$$

We therefore arrive at the expression

$$C_{n-2}^{0} + C_{n-2}^{1} + C_{n-3}^{2} + \cdots$$

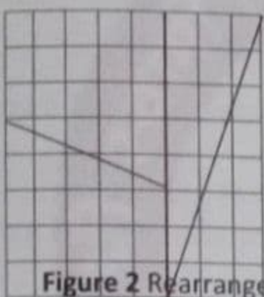
$$= C_{n-1}^{0} + C_{n-2}^{1} + C_{n-3}^{2} + \cdots$$

to represent the sum of terms of the nth rising diagonal of Pascal's triangle. Indeed, if we look at diagram (4) of Pascal's triangle, this corresponds to the correct expression. Thus, as we know the first two diagonals are both 1, and we now see that the sum of all numbers in the $(n-1)^{st}$ diagonal plus the sum of all numbers in the $(n-2)^{nd}$ diagonal is equal to the sum of the nth diagonal, we have proved that the sum of terms on the nth diagonal is always equivalent to the n^{th} Fibonacci number.

Game about Fibonacci number

Game:

Cut a 8×8 square into four parts(as figure 1 shows), and rearrange the four parts into a new 5×13 rectangle as figure 2 shows



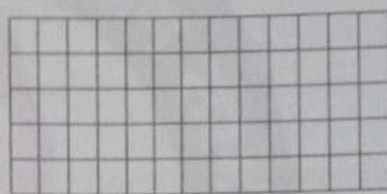


Figure 2 Rearrange into a new rectangle

Figure 1 Original square

When we calculate the area of figure 1, we can easily get that the area of the square equals to 64.However, the area of the new rectangle equals to 65.During this process, we have not abandoned or added any piece of paper into the new rectangle, so the area of the original square and the rectangle should equal to each other. In order to figure out the reason for the area change and the hidden mathematical principle behind, we picked this game as our topic

Mathematic principle behind

We observed the square carefully and found out that the length of the sides of the square and the rectangle are all Fibonacci numbers.We have

n=6

$$a_n = a_6 = 8,\ a_{n-1} = a_5 = 5,\ and\ a_{n+1} = a_7 = 13$$

In fact, Fibonacci sequence has the following property:

$$a_{n+1}a_{n-1} = a_n^2 + (-1)^n$$

In our game, the product of $a_{n+1}a_{n-1}$actually represents the area of the rectangle $S_{rectangler}$, and a_n^2 represents the area of the squareS_{square}.

$$S_{rectangle} = a_7 a_5 = 5 * 13 = 65$$

$$S_{square} + (-1)^6 = a_6^2 + (-1)^6 = 64 + (-1)^6 = 65$$

If we observe the new rectangle carefully, we will find there is a gap on the rectangle. So the area of the original square has never changed. The area change only caused by adding extra part into the area of the rectangle.

We use the mathematical induction method to prove this property:

Proof: Step 1: Let n=1, $a_0 = 0,\ a_1 = 1,\ a_1 = 1$

Then equality(1)holds.

Step 2 Suppose it is true for n=k. Then we get the equality

$$a_{k-1}a_{k+1} = a_k^2 + (-1)^k$$

Step 3 Now we show it is true for n=k+1.

According to the definition if the Fibonacci sequence, we have the following recursion formula:

$$a_{k+1} = a_k + a_{k-1};$$

$$a_{k+2} = a_{k+1} + a_k = 2a_k + a_{k-1}$$

Then we get $a_{k-1}(a_k + a_{k+1}) = a_k^2 + (-1)^k$

$$a_k^2 - a_{k-1}a_k - a_{k-1}^2 = (-1)^{k+1}$$

$$a_k(2a_k + a_{k-1}) = a_k^2 + 2a_k a_{k-1} + a_{k-1}^2 + (-1)^{k+1}$$

Thus

$$a_k a_{k+2} = a_{k+1}^2 + (-1)^{k+1}$$

It is true for n=k+1.

Thus property (1) is proved.

Another situation

When we reshape the four parts of the original square in another way as figure 3 shows, we get a new polygon. The intersection in the new polygon changes its area into 63. We can also use another property of Fibonacci sequence to explain why the change happens.

Figure 3: Rearrange four parts into a new polygon

The Fibonacci sequence also has the following

$$4a_{n-1}a_{n-2} + a_{n-2}a_{n-4} = a_n^2 + (-1)^{n-1}$$

Proof: For we have (2) $a_n a_{n+2} = a_{n+1}^2 + (-1)^{n+1}$

$$4a_{n-1}a_{n-2} + a_{n-2}a_{n-4} = 4a_{n-1}a_{n-2} + a_{n-2}(2a_{n-2} - a_{n-1})$$

$$= a_{n-2}(3a_{n-1} + 2a_{n-2})$$
$$= a_{n-2}(a_n + a_{n-2} + 2a_{n-1})$$
$$= a_{n-1}^2 + (-1)^{n-1} + a_{n-2}^2 + 2a_{n-2}a_{n-1}$$
$$= a_n^2 + (-1)^{n-1}$$

Thus the property is proved.

When we calculate the area of the new polygon, we actually do not involve the intersection part. So the area of the new polygon we get is smaller than the original square.

Perfect situation

After we had researched two situations above, it is reasonable to think about the question that how can we cut the original square to get a perfect rectangle.

Suppose the length of the sides are x and y as figure 4 shows, and we can get a new rectangle without any gap or intersection.4

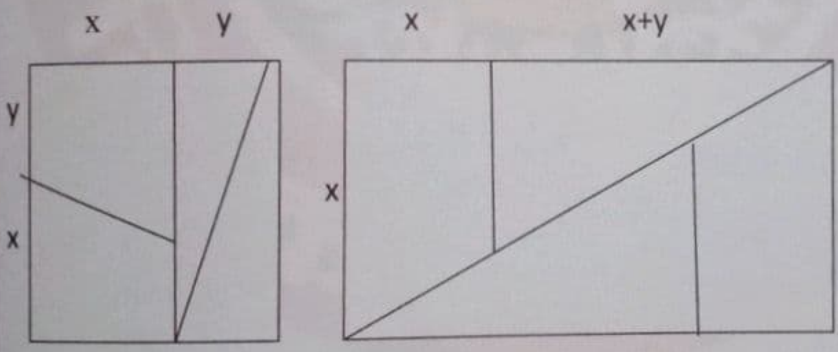


Figure 4 The situation we can get a perfect rectangle

Suppose the area of the original square is $SSquare$ and the area of rectangle is $S_{rectangle}$. Then we have

$$S_{square} = (x + y)$$

$$S_{rectangle} = (2x + y)x$$

Let

$$S_{square} = S_{rectangle}$$

Then we have

$$(x+y)^2 = (2x+y)x$$

i.e.

$$(x|y)^2 - (x|y) \ -1 = 0$$

Thus we get the solution

$$\frac{x}{y} = \frac{1 \pm \sqrt{5}}{2}$$

Because x and y are the length of the sides, so we only take the positive one. It is easy to realize that

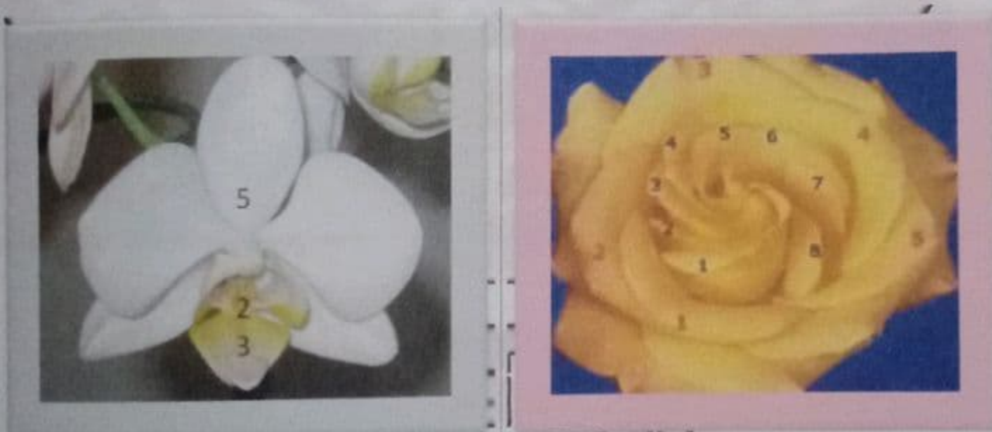
$$\frac{y}{x} = \frac{2}{1+\sqrt{5}} \approx 0.618$$

In fact, if y equals to a_{n-2} and x equals to $a_{n-1}$the proportion of x for y represents the proportion of a_{n-1} for a_{n-2}. When n tends to infinity, the ratio tends to be0.618 (golden ratio).So the perfect situation above happened when x equals to a_{n-1}, y equals to a_{n-2} and n tends to be infinity.

Applications

Application in botany

Figure5 The Fibonacci spiral appeared in some kind of aloes



Considering the number of the petals of the flowers, some of them follow Fibonacci numbers. One possible reason for why this phenomenon happens is that they try to decrease the overlapped area to get more sunlight.

The hidden spiral in flowers

According to figure when we count the number of its spirals, we will find there are 13counterclockwise spiral sand 21 clockwise spirals. These two numbers are both Fibonacci numbers .One possible reason for the Fibonacci spiral appears in the sunflower is to fully use the space.

Botanists also find the numbers of branches of trees are always Fibonacci number. They find that after a certain period of time, each old branch of a tree will get a new one and need one more period of time to turn to an old one. If the tree has only one branch at very beginning, after a year it will have two branches and in next circle, it has three branches. Every year the total branches of the tree composed a Fibonacci sequence. One possible reason for it is that every new branch needs one year to get mature.

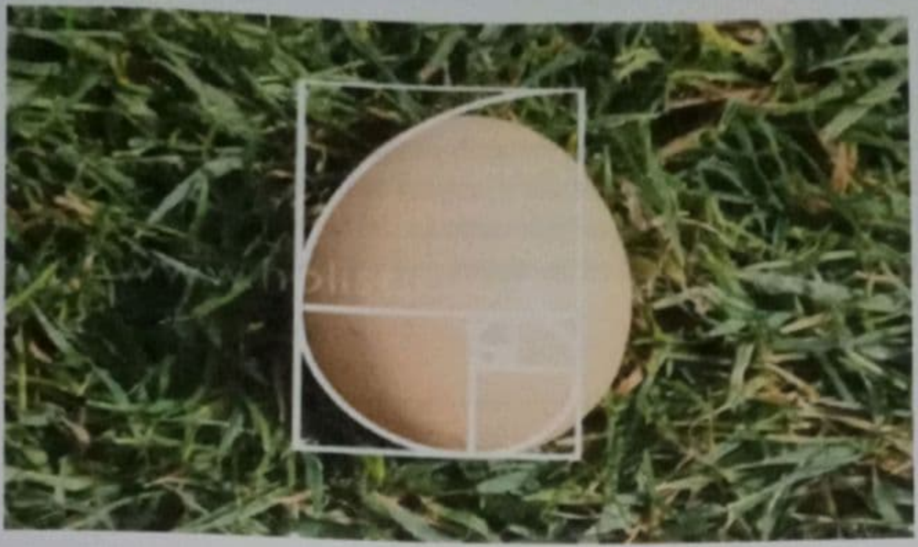
Since Fibonacci numbers shows in some of the flower petal numbers and the arrangements of the inflorescence, many plants do not show any

Fibonacci number in their arrangements. In fact, scientists have not figured out the pattern accurately the plants arrange their leaves and flowers. We can only assume that plants apply Fibonacci numbers and the golden ratio in their growth to make their space utilizations more effectively. However, it is also possible that plants do not understand Fibonacci sequence at all, and may be they just grow to the shape they like.

Application in zoology

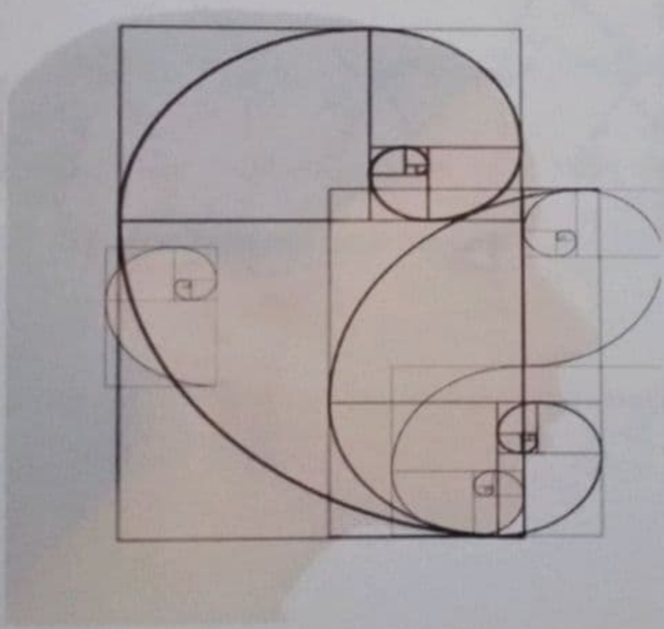
The hidden Fibonacci spiral in American Giant Millipede, Monarch caterpillar, Panther Chameleon, Snails and Fingerprints

Application in composition



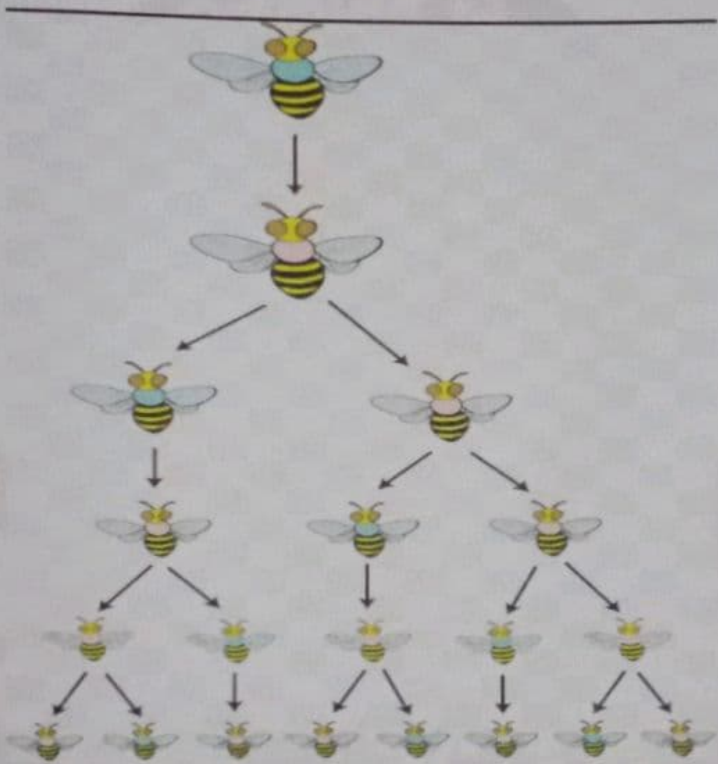
The photographer use the spiral into composition to highlight the rose and chicken egg.

The following figure is a stage photo picked from BBC's show Sherlock. The starting point of Fibonacci spiral in figure highlights parts of girl's face. The application of Fibonacci spiral can make the picture looked more harmonious



Application in bees

The breed form of bees is very special, for the male bees are all hatched from unfertilized eggs and the female ones are all hatched from fertilized one. Hence the male bees are only the fathers of the female ones, and male bees also have no father at all. When scientists observed the family tree of the bees, they found that the number G_n of the n generation of the first male bee equals to the a_n in Fibonacci sequence.



The family tree of male bee

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Thank You