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प्रति,
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क्रांतीसिंह नाना पाटील महाविद्यालय,
वाळवा.

विषय : इंटरनॅशनल कॉन्फरन्ससाठी विद्यार्थी पाठविलेबाबत. . .

महोदय,

वरील विषयास अनुसरुन आपण आपल्या महाविद्यालयात दिनांक ६ व ७ मार्च २०२६ रोजी "Changing Scenaria in Science and Technology: A Multidisciplinaryj Approach (ICCSST-२०२६)" या विषयावर आयोजित केलेल्या इंटरनॅशनल कॉन्फरन्ससाठी खालील प्राध्यापक व एम.एस्सी.मॅथस्.भाग-२ मधील विद्यार्थ्यांना पाठवित आहोत. कृपया त्यांना सहभागी करुन घ्यावे ही विनंती.

१) प्रा.श्वेता कोष्टी
२) कु.पल्लवी गुडले
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६) कु.निशिगंधा भोसले
७) कु.नेहा शिंदे

कळावे,

आपला विश्वासू,

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प्र. प्राचार्य
विवेकानंद कॉलेज, कोल्हापूर
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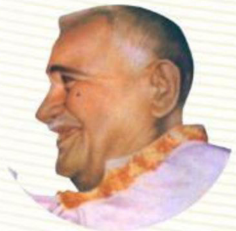

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This is to certify that **Miss. Pallavi Bhujgonda Gudle**

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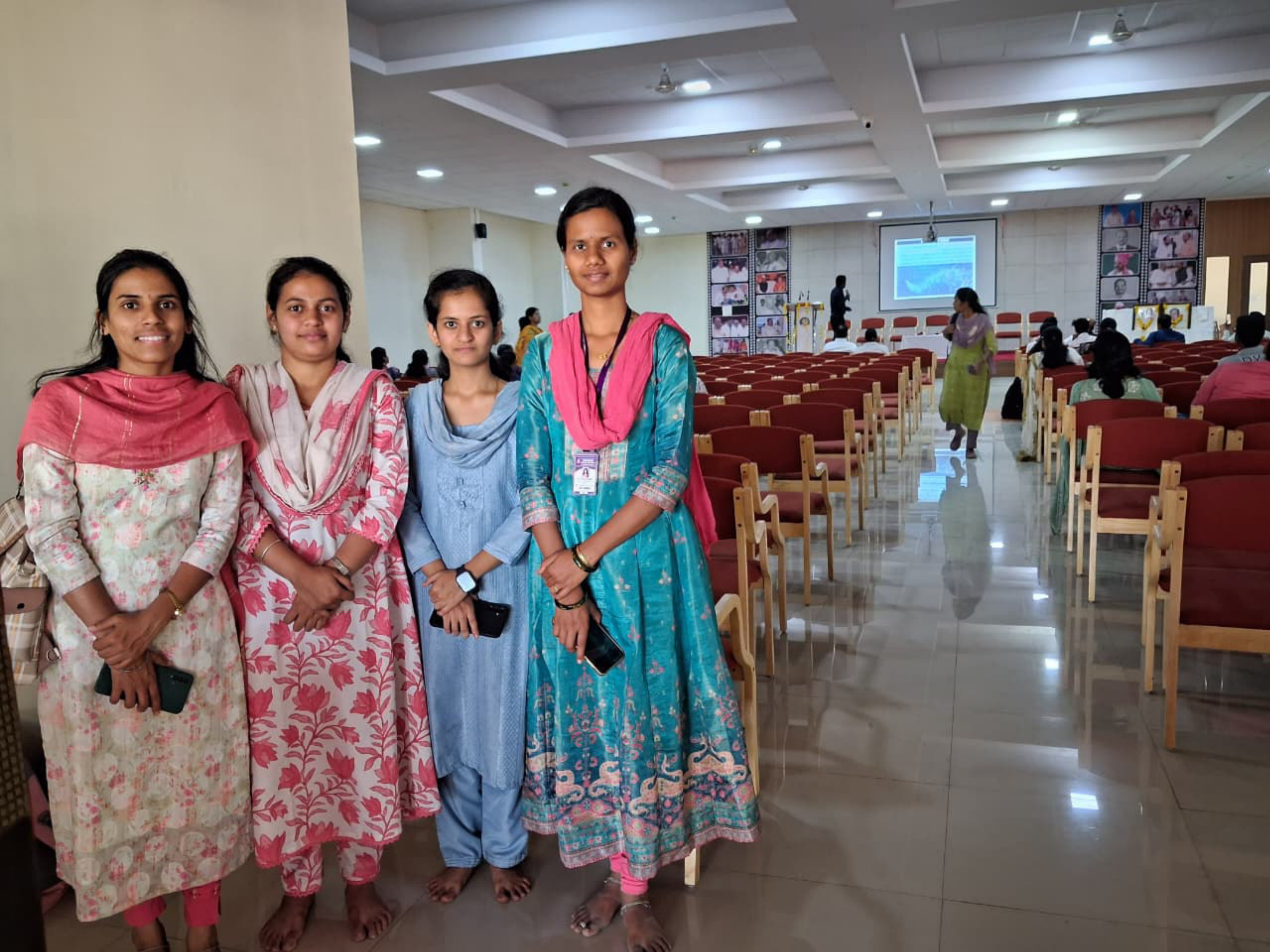

NUMERICAL SOLUTIONS OF FUZZY INTEGRAL EQUATIONS

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ABSTRACT

In this paper, we apply the Adomian Decomposition Method (ADM) to obtain approximate analytical solutions of fuzzy Volterra and Fredholm integral equations. By employing the α-cut representation of fuzzy numbers, the fuzzy problems are transformed into systems of two crisp Volterra and Fredholm integral equations (FIE), respectively. The ADM is then applied to each resulting equation separately, leading to convergent series solutions for the fuzzy unknown functions. The effectiveness and accuracy of the proposed approach are illustrated through representative examples.

Keywords: Integral equation, Adomain decomposition method, Fuzzy variables, $\propto -cuts,$ fuzzy integral equation,

I. INTRODUCTION

Integral equations originated in the late eighteenth and early nineteenth centuries, with early contributions attributed to Niels Henrik Abel in 1824[1]. The theory was later systematically developed by Vito Volterra, who introduced Volterra integral equations around 1896, and by Ivar Fredholm, who formulated the Fredholm integral equation theory in 1900 [8]. Since then, integral equations have become essential tools in applied mathematics and appear naturally in many scientific and engineering disciplines, including continuum mechanics, potential theory, fluid dynamics, quantum mechanics, heat transfer, population dynamics, and control theory.

The development of fuzzy integral equations followed the introduction of fuzzy set theory by Lotfi A. Zadeh in 1965 [2]. Although there is no single inventor of fuzzy Volterra or fuzzy Fredholm integral equations, significant contributions to fuzzy analysis were made by Didier Dubois and Henri Prade, while Olli Kaleva played a key role in the development of fuzzy differential equations in the late 1980s [3] Based on these foundational works, fuzzy Fredholm and fuzzy Volterra integral equations were formally studied during the 1990s [9].

In recent decades, fuzzy integral equations have attracted increasing interest due to their wide applicability in modeling systems with uncertainty, particularly in fuzzy control and decision-making problems. Several notions of fuzzy integration have been proposed, including approaches based on the extension principle, Riemann-type integrals, and Lebesgue-type integrals. One of the earliest applications of fuzzy integral equations involved the study of fuzzy Fredholm integral equations of the second kind, followed by extensive research on numerical and semi-analytical solution techniques.

Various approximate and numerical methods have been developed to solve fuzzy Volterra and Fredholm integral equations, such as the Adomian decomposition method (ADM), variational iteration method, homotopy analysis method, regularization techniques, and wavelet-based methods [10-13]. Among these, the ADM has proven to be particularly effective due to its simplicity, rapid convergence, and ability to handle nonlinearities without linearization or discretization.

This paper basically focused on the recent advances in approximate methods for solving fuzzy Volterra integral equations and fuzzy Fredholm integral equations (FIE) of the second kind, with particular emphasis on the ADM and its modified forms.

II. PRELIMINARIES

1) Fuzzy set theory:

A fuzzy set A of a set X is a function A:X→ I. The fuzzy set A can also be defined as

$A = \{(x, A(x)), x \in X, A(x) \in [0,1]\}$ Here, f(x) is called the grade of membership of x.

2) α-cut of Fuzzy Set:

2.1 The crisp set ${}^{\alpha}A = \{x \epsilon X | A(x) \geq \alpha\}$ is called α- level cut or α – cut of a fuzzy setA.

2.2 Support of a fuzzy set A is the crisp set ${}^{0+}A = \{x \epsilon X | A(x) > 0\}$

2.3 Core of a fuzzy set A is the crisp set A=$\{x \epsilon X | A(x) = 1\}$

2.4 The fuzzy set A is called normal if h(A)=1.

2.5 The fuzzy set A is called subnormal if h(A)<1

3)Fuzzy number

A fuzzy number A is a fuzzy subset of the set of real numbers $\mathbb{R}$ such that

(i) A must be a normal fuzzy set;

(ii) ${}^{\alpha}A$ must be a closed interval for every $x \in [0,1]$

(iii) The support of A, ${}^{0+}A$, must be bounded.

4) Triangular Fuzzy Number:

The triangular fuzzy number $r_{(\varepsilon,\delta)}$can be represented as $(r_{\varepsilon,\delta})_\alpha = [r - \varepsilon(1-\alpha), r + \delta(1-\alpha)]$.

5) Interval Arithmetic:

We know that an interval $[a_1, a_2]$ is a subset of R such that $A = [a_1, a_2] = \{t \epsilon \mathbb{R} : a_1 \leq t \leq a_2, a_1; a_2 \epsilon \mathbb{R}\}$

If $A = [a_1, a_2]$ and $B = [a_1, a_2]$ are two intervals, then the arithmetic operations on intervals are defined as follows:

Addition: $[a_1, a_2] + [b_1, b_2] = [a_1 + b_1, a_2 + b_2]$

Subtraction: $[a_1, a_2] - [b_1, b_2] = [a_1 - b_2, a_2 - b_1]$

Product: $[a_1, a_2].[a_1, a_2]$=$[\min(a_1b_1, a_1b_2, a_2b_1, a_2b_2), max(a_1b_2, a_1b_1, a_2b_1, a_2b_2)]$

Division: $[a_1, a_2]/[a_1, a_2]$=$[\min(a_1/b_1, a_1/b_2, a_2/b_1, a_2/b_2), max(a_1/b_1, a_1/b_2, a_2/b_1, a_2/b_2)]$

6) Integral Equation:

An integral equation is an equation in which the unknown function appears under an integral sign and is related to a known function through an integral operator.

6.1 Volterra Integral Equation (VIE):

A Volterra integral equation is an integral equation in which the upper limit of integration is a variable.

$u(t) = f(x) + \lambda \int_a^s k(x,t)u(t)dt.$

6.2 Fredholm Integral Equation (FIE):

A Fredholm integral equation is an integral equation in which both limits of integrations are fixed constant.

$u(t) = f(x) + \lambda \int_a^b k(x,t)u(t)dt$.

7) Fuzzy Integral Equation:

A fuzzy integral equation is a mathematical equation used to model system containing uncertainty, vagueness, or imprecise information, where the unknown function or coefficients are expressed as fuzzy numbers.

7.1 Fuzzy Volterra Integral Equation (FVIE):

A Volterra integral equation is an integral equation in which the upper limit of integration is a variable.

$\widetilde{u(t)} = \widetilde{f(x)} + \lambda \int_a^s k(x,t).u(t)dt.$

7.2 Fuzzy Fredholm Integral Equation (FFIE):

A Fredholm integral equation is an integral equation in which both limits of integrations are fixed constants.

$\widetilde{u(t)} = \widetilde{f(x)} + \lambda \int_a^b k(x,t).u(t)dt$

III. SOLUTION OF FUZZY INTEGRAL EQUATION (FIE)

3.1 Solution of Fuzzy Volterra Integral Equation

Here, we solving integral equation by ADM. This method provides a solution of integral equation in the form of series. In ADM we usually expressed the solution u(x) in the form of series $\sum_{n=0}^{k} u_n(x)$. It is important to note that the series obtain from u(x) frequently provides exact solution in closed form. However in some problems the series from u(x) cannot be evaluated then the truncated series $\sum_{n=0}^{k} u_n(x)$, $k \in \mathbb{N}$ is usually used to n=0 approximate solution u(x).

A Fuzzy Volterra integral equation is an integral equation where the unknown function, kernel, or input data are defined by fuzzy sets rather than crisp numbers.

Consider, the FVIE of second kind

$\widetilde{u(x)} = \widetilde{f(x)} + \lambda \int_a^s k(x,t)\, \widetilde{u(t)}dt$ .... (1)

where, $\widetilde{u(x)}$ is a unknown function,

$\widetilde{f(x)}$ is a known function,

K(x,t) is kernel, $a \leq t \leq s.$

Here, we convert fuzzy functions into $\alpha - cut's$ by

$[\underline{u(x,\alpha)};\ \overline{u(x,\alpha)}] = [\underline{f(x,\alpha)};\ \overline{f(x,\alpha)}] + \lambda \int_a^s k(x,t)\,[\underline{u(t)}, \overline{u(t)}]dt$

Equating lower and upper cut's of fuzzy equation we get the two crisp equation as follow:

$$\underline{u(x,\alpha)} = \underline{f(x,\alpha)} + \lambda \int_a^s k(x,t)\,.\underline{u(t)}dt$$

$$\overline{u(x,\alpha)} = \overline{f(x,\alpha)}] + \lambda \int_a^s k(x,t)\,\overline{u(t)}dt$$

Above both crisp integral equations can be solve by ADM.

Thus $u(x,\alpha) = \sum_{n=0}^{\infty} u_n\,(x,\alpha)$ is the solution of integral equation (1)

Using above equation we get crisp equation as

$$\sum_{n=0}^{\infty} \underline{u_n\,(x,\alpha)} = f(x) + \lambda \int_a^s \sum_{n=0}^{\infty} k(x,t)\,\underline{u_n(t)}dt$$

$\underline{u_0\,(x,\alpha)} = f(x),$

$\underline{u_1\,(x,\alpha)} = \lambda \int_a^s k(x,t)\,.\underline{u_1(t)}dt,$

$\underline{u_2\,(x,\alpha)} = \lambda \int_a^s k(x,t)\,.\underline{u_2(t)}dt,$

:

$\underline{u_n\,(x,\alpha)} = \lambda \int_a^s k(x,t)\,.\underline{u_{n-1}(t)}dt \quad \ldots n \geq 1.$

This equations can be written in recursive manner as follows.

$$\therefore \underline{u_{n+1}(x,\alpha)} = \lambda \int_a^s k(x,t)\,.\underline{u_n(t)}dt$$

Simillarly we can solve equation

$\therefore \overline{u_{n+1}(x,\alpha)} = \lambda \int_a^s k(x,t)\ \overline{u_n(t)}\,dt$

Finally the solution $u\,(x,\alpha)$ of integral equation (1) can be obtain by above two equations.

$[\underline{u_{n+1}(x,\alpha)};\ \overline{u_{n+1}(x,\alpha)}] = \int_a^s k(x,t)[\underline{u_n}\ ,\overline{u_n}\ ]$

3.2 Solution of Fuzzy Fredholm Integral Equation:

A FFIE is similar to a Fuzzy Volterra integral equation, but the upper limit of integration is constant instead of variable.

$\widetilde{u(x)} = \widetilde{f(x)} + \lambda \int_a^b k(x,t)\,\widetilde{u(t)}$dt

where, $\widetilde{u(x)}$ is a unknown function,

$\widetilde{f(x)}$ is a known function,

K(x,t) is kernel, & a and b both are constant.

IV. NUMERICAL SOLUTION:

1) Find the solution of $\widetilde{u(\varepsilon,\delta)} = 1_{1,1} + \int_0^x xt\, \widetilde{u(t_{\varepsilon,\delta})}dt$ where $\widetilde{u(\varepsilon,\delta)}$ *and* $\widetilde{u(\varepsilon,\delta)}$ are unknown functions.

Let, $u_{(\varepsilon,\delta)} = 1_{1,1} + \int_0^x xt\, u(t_{\varepsilon,\delta})dt$ …….. (1)

$\bar{u}\,(x) = 1_{(1,1)} + \int_0^x xt\, u(t_{\varepsilon,\delta})dt = [\alpha; 2-\alpha] + \int_0^x xt\, u(t_{\varepsilon,\delta})dt$

Equating the lower and upper $\alpha -$ cuts we get the two crisp equation as follow:

$\underline{u}(x) = [\alpha] + \int_0^x xt\, \underline{u(t)}\, dt.$ …….. (2)

$\overline{u}(x) = [2-\alpha] + \int_0^x xt\, \overline{u}(t)\, dt.$ …….. (3)

Solving this two Volterra Integral equation by ADM.

Thus $\underline{u_{(\varepsilon,\delta)}} = \sum_{n=0}^{\infty} \underline{u_{(\varepsilon,\delta)}}$ is solution of VIE (2) is,

$\underline{u_0\,(\varepsilon,\delta)} = \alpha$

$$\underline{u_1\,(\varepsilon,\delta)} = \int_0^x x.t.\underline{u_0}\, dt = \int_0^x x.t.\alpha\, dt = x\alpha\frac{x^2}{2} = \alpha\frac{x^3}{2}$$

$$\underline{u_2\,(\varepsilon,\delta)} = \int_0^x x.t.\underline{u_1}\, dt = \int_0^x x.t\,\alpha\frac{t^3}{2}\, dt = \alpha\frac{x^6}{10}$$

$$\underline{u_3\,(\varepsilon,\delta)} = \int_0^x x.t.\underline{u_2}\, dt = \int_0^x x.t\alpha\frac{t^6}{10}\, dt = \alpha\frac{x^9}{80}$$

$$\underline{u_{(\varepsilon,\delta)}} = \sum_{n=0}^{\infty} \underline{u_{(\varepsilon,\delta)}} \Rightarrow \underline{u(\varepsilon,\delta)} = \alpha\left(1+\frac{x^3}{2}+\frac{x^6}{10}+\frac{x^9}{80}+\cdots\right)$$

Simillarly, we get

$\overline{u(\varepsilon,\delta)} = (2-\alpha)\left(1+\frac{x^3}{2}+\frac{x^6}{10}+\frac{x^9}{80}+\cdots\right).$

Thus solution of fuzzy VIE is

$$\left[\underline{u(\varepsilon,\delta)}, \overline{u(\varepsilon,\delta)}\right] = \left[\alpha\left(1+\frac{x^3}{2}+\frac{x^6}{10}+\frac{x^9}{80}+\cdots\right), (2-\alpha)\left(1+\frac{x^3}{2}+\frac{x^6}{10}+\frac{x^9}{80}+\cdots\right)\right]$$

2) Find the solution of $\bar{u}\,(x) = 9_{(2,3)}x^2 + \int_0^1 \frac{x^2t^2}{2}\overline{u_0}\, dt$ where $\overline{u(x)}, \overline{u(t)}$ are fuzzy unknown function.

Convert the Fuzzy equation into $\alpha - cut$, we get

$$\bar{u}\,(x) = [7+2\alpha; 12-3\alpha]x^2 + \int_0^1 \frac{x^2t^2}{2}\overline{u(t)}\, dt\,.$$

After equating the lower and upper part of the equation we get two crisp equation as follow:

$\underline{(u\,)}\,(x) = [7+2\alpha]x^2 + \int_0^1 \frac{x^2t^2}{2}\underline{u}(t)dt\,.$ …….. (1)

$\bar{u}\,(x) = [12-3\alpha]x^2 + \int_0^1 \frac{x^2t^2}{2}\overline{u(t)}\, dt\,.$ …….. (2)

Solving above two FIE by ADM:

Let $u_{(\varepsilon,\delta)} = \sum_{n=0}^{\infty} u_{(\varepsilon,\delta)}$ be the solution of above two equations.

Thus, $\underline{u_0(\varepsilon,\delta)}$= (7+2$\alpha$) x^2

$\underline{u_1(\varepsilon,\delta)} = \int_0^x \frac{x^2t^2}{2}\underline{u_0}\, dt = \frac{x^7}{10}(7+2\alpha)$

$\underline{u_2}(\varepsilon,\delta)=\int_0^x \frac{x^2t^2}{2}\underline{u_1}\,dt=\frac{x^{12}(7+2\alpha)}{200}$

$\underline{u_3}(\varepsilon,\delta)=\int_0^x \frac{x^2t^2}{2}\underline{u_2}\,dt=\frac{x^{17}(7+2\alpha)}{6000}$

Thus solution of equation (1) is

$\underline{u}(\varepsilon,\delta)=x^7(7+2\alpha)\left[\frac{1}{10}+\frac{x^{12}}{200}+\frac{x^{17}}{6000}+\cdots\right]$

Similarly we find the solution of equation two, we have

$\overline{u}(\varepsilon,\delta)=x^7(12-3\alpha)\left[\frac{1}{10}+\frac{x^{12}}{200}+\frac{x^{17}}{6000}+\cdots\right].$

The solution of fuzzy Fredholm Integral equation is

$\left[\underline{u_3}(\varepsilon,\delta),\overline{u(\varepsilon,\delta)}\right]=\left[x^7(7+2\alpha)\left[\frac{1}{10}+\frac{x^{12}}{200}+\frac{x^{17}}{6000}+\cdots\right],x^7(12-3\alpha)\left[\frac{1}{10}+\frac{x^{12}}{200}+\frac{x^{17}}{6000}+\cdots\right]\right]$

V. CONCLUSION

In this paper, the Adomian Decomposition Method (ADM) was successfully applied to obtain approximate analytical solutions of fuzzy Volterra and Fredholm integral equations. By using the α-cut representation of fuzzy numbers, the original fuzzy problems were effectively transformed into systems of two crisp Volterra and Fredholm integral equations. The ADM was then implemented for each resulting equation independently, yielding convergent series solutions for the fuzzy unknown functions. The results demonstrate that the proposed approach is accurate and efficient, and that ADM is well suited for solving fuzzy integral equations in the presence of uncertainty.

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